

# Effective Grushko Decompositions

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June 21, 2009

## Abstract

We show how to find a Grushko decomposition of a finitely presented group, given a solution to its word problem.

## 1 Introduction

Given a finitely generated group  $G$  it is possible to represent it as a free product

$$G = H_1 * \dots * H_k * F_n$$

where all the  $H_i$  are freely indecomposable and  $F_n$  is a free group of rank  $n$ . Such a decomposition is called a Grushko decomposition and it is *universal* in the sense that for any other such free decomposition

$$G = K_1 * \dots * K_l * F_m$$

we must have that  $k = l, m = n$  and that each  $K_i$  is conjugate in  $G$  to a unique  $H_j$ .

The aim of this paper is to prove the following theorem:

**Theorem 1.1.** *Given a group presentation  $G = \langle X \mid R \rangle$  and a solution to the word problem in this presentation there is an algorithm that decides if  $G$  admits a free decomposition*

$$G = G' * G''$$

*and if so, produces presentations for the factors  $G', G''$  and gives explicit injections*

$$G' \hookrightarrow G \hookleftarrow G''$$

By induction this yields:

**Corollary 1.2.** *Given a group presentation  $G = \langle X \mid R \rangle$  and a solution to the word problem in this presentation, there is an algorithm that produces a Grushko decomposition of  $G$ .*

In a sense this is the strongest result possible: the restrictions on the input are as minimal as can be reasonably expected. This result also extends all previously known results, which we now briefly survey.

Diao and Feighn in [DF05] showed how to find a Grushko decomposition of a fundamental group of a graph of free groups. Their techniques rely on Whitehead methods refined by Gersten and group actions on square complexes. Kharlampovich and Miasnikov in [KM05b] showed how to find a Grushko decomposition of a fully residually free group by running their Elimination Process: the free decomposition becomes apparent by “separating the variables” in the defining equations. Finally, Dahmani and Groves in [DG08] extended the previous result to the class of toral relatively hyperbolic groups by generalizing an unpublished algorithm for hyperbolic groups due to Gerasimov. Their approach was to decide some connectivity criterion of the boundary of toral relatively hyperbolic groups.

Our approach is to give an algorithm that decides whether or not it is possible to draw a *pattern* in the canonical cell complex corresponding to our group presentation which represents a nontrivial free splitting. Patterns (or tracks) were first used Dunwoody in [Dun85], and are very useful tools to study the actions of finitely presented groups on trees.

To decide if such a pattern exists we will use a variant of the Elimination Process, an algorithm invented by Razborov and refined by Kharlampovich and Myasnikov that was originally designed to describe solutions of equations into a free group. Aside from the algorithm itself, which is quite involved, we only use Bass-Serre theory and basic topology of cell complexes.

## 1.1 Acknowledgements

I wish to thank Olga Kharlampovich for her answers to some very technical questions and Alexei Miasnikov for his numerous attempts (and success) to explain the elimination process in advanced classes and for drawing my attention to tracks. I am also grateful to Ilya Kazachkov and Montserrat Casals-Ruiz for discussions.

## 2 Preliminaries

Let  $\langle X \mid R \rangle$  be a finite presentation of a group  $G$ . Let  $(\mathcal{C}, x_0)$  be the canonical based polygonal 2-complex associated to  $\langle X \mid R \rangle$ . We identify  $G = \pi_1(\mathcal{C}, x_0)$ .

### 2.1 Patterns, tracks, splittings

These next notions are taken from [DS99]

**Definition 2.1.** Let  $X$  be a 2 complex. A *pattern*  $P \subset X$  is an embedded 1-complex such that for every 2-cell  $D \subset X$ ,  $P \cap D$  is a (possibly empty) collection of closed arcs, the endpoints of these arcs do not coincide with 0-cells of  $X$ , and the interiors of these arcs lie in the interior of  $D$ .

**Definition 2.2.** A connected component of a pattern is called a *track*. If a track has a regular neighbourhood  $t \subset N(t) \subset X$  such that  $N(t) \simeq t \times [0, 1]$  then it is called *two-sided*.

**Definition 2.3.** Let  $Y \subset X$  be a topological subspace. The inclusion map  $i : Y \subset X$  induces a homomorphism  $i_\# : \pi_1(Y) \rightarrow \pi_1(X)$  of fundamental groups which is well defined up to conjugacy. Given a basepoint  $x_0$  of  $X$  and  $\pi_1(X, x_0)$ , we will denote by  $Gp(Y)$  a subgroup of  $\pi_1(X, x_0)$  that lies in the conjugacy class of  $i_\#(\pi_1(Y))$ . We will also call this the *subgroup induced by  $Y$* .

**Proposition 2.4.** For a two sided track  $t$  contained in a 2 complex  $X$ , the union

$$X = (X - \text{int}(N(t))) \cup N(t)$$

induces a one edged splitting of  $\pi_1(X, x_0)$  whose edge group is conjugate to  $Gp(t)$ .

*Proof.* By definition of a track we may identify  $N(t)$  with  $t \times [-1, 1]$ . Now, adding countably many 2-cells to  $N(t)$  and  $X - (t \times (-1, 1))$  so as to ensure  $\pi_1$ -injectivity of

$$X - (t \times (-1, 1)) \subset X \supset t \times [-1, 1]$$

we get, via the attaching maps  $t \times \{1\} \hookrightarrow X - (t \times (-1, 1))$  and  $t \times \{-1\} \hookrightarrow X - (t \times (-1, 1))$  a decomposition of  $\pi_1(X, x_0)$  as the fundamental group of a graph of spaces with edge group conjugate to  $Gp(t)$ .  $\square$

## 2.2 Resolutions

Suppose we have a group  $G = \pi_1(\mathcal{C}, x_0)$  that on a tree real tree  $T$ , then we can construct a map  $\phi : (\tilde{\mathcal{C}}, \tilde{x}_0) \rightarrow T$  that is  $G$ -equivariant, i.e. for  $y \in \tilde{\mathcal{C}}$ ,  $g * \phi(y) = \phi(g * y)$  where  $G$  acts on  $(\tilde{\mathcal{C}}, \tilde{x}_0)$  by Deck transformations. We describe the construction. We assume that  $G$  acts transitively on the vertices of  $\tilde{X}$ . We proceed as follows:

1. First map  $\tilde{x}_0$  to some  $t_0 \in T$  and extend this map to  $\tilde{X}^0$  by defining

$$\phi(g * x_0) := g * t_0$$

2. There is a correspondence between the vertices of  $\tilde{\mathcal{C}}$  and  $G$ , and at each vertex each adjacent edge corresponds to one of the generators  $g_1^{\pm 1}, \dots, g_m^{\pm 1}$ . We then extend the map  $\phi$  to  $\tilde{\mathcal{C}}^1$  by linearly mapping the edge  $e$  from  $v$  to  $g_i^{\pm 1}v$  (which is homeomorphic to  $[0, 1]$ ) to the segment  $[\phi(v), \phi(g_i^{\pm 1}v)] \subset (T)$ .
3. We finally extend  $\phi$  to all of  $\tilde{\mathcal{C}}$ . For a 2-cell  $D$  whose boundary is consists of the edges  $e_1, \dots, e_n$ , we consider the regular Euclidean  $n$ -gon  $\mathbb{E}(D)$ , the natural homeomorphism  $D \simeq \mathbb{E}(D)$  gives  $D$  a Euclidean structure. We have that  $\phi$  is defined on  $\partial(D)$ . For any points  $u, v \in \partial(D)$  such that  $\phi(u) = \phi(v)$  we map the points in  $D$  that lie on the line segment  $[u, v]$  to  $\phi(u) = \phi(v)$ . Any point  $y$  in  $D$  that does not lie in in such a line segment lies inside a  $k$ -gon  $D' \subset D$  that is bounded by lines  $l_1, \dots, l_k$  that are sent to some point  $t \in T$  via  $\phi$ , we define  $\phi(y) = t$ .

**Definition 2.5.** The map we described is called a *resolution*

The following is proved in Lemmas 2.1 and 2.2 of [DS99].

**Proposition 2.6.** *Suppose a group  $G = \pi_1(\mathcal{C}, x_0)$  acts on a simplicial tree  $T$  without edge inversions, then we can obtain a pattern  $P \leq \mathcal{C}$  whose tracks  $\tau_i$  are two sided called a resolving pattern. Each lift of a track  $\tau_i$  in the universal cover separates  $\tilde{\mathcal{C}}$  into two components. The dual graph of the lifts of  $P$  in  $\tilde{\mathcal{C}}$  is also a  $G$ -tree  $T'$  and we have a  $G$ -equivariant morphism of trees  $T' \rightarrow T$ .*

*sketch.* We give a sketch of the first claim. The reader can fill in the details by considering the construction of  $\phi$  more closely. Let  $\phi$  be the resolution and consider the preimage of  $(G$ -equivariant) edge midpoints. We see that this preimage is a subgraph of  $\mathcal{C}$ , moreover this maps to a 1-subcomplex  $P \leq X$  via the quotient map  $\mathcal{C} \rightarrow G \backslash \mathcal{C} = \mathcal{C}$ . Because the action of  $G$  on  $T$  was without edge inversions, we see that  $P$  has two sided tracks.

The rest is proved in Theorem Section 2 of [DS99].  $\square$

**Definition 2.7.** Let  $\mathcal{C}$  be a cell complex and let  $P \subset \mathcal{C}$  be a two sided pattern. We denote by  $T(P, \mathcal{C})$  the dual graph of the lifts of  $P$  in  $\tilde{\mathcal{C}}$ . This is the tree given in Proposition 2.6.

It is important to note that the induced subgroups of resolving patterns are always finitely generated. Therefore, given an action of  $G$  on a simplicial tree  $T$ , we can think of a resolving pattern and the dual tree  $T'$  as an approximation of the action of  $G$  on  $T$  such that edge stabilizers are finitely generated.

**Definition 2.8.** A track  $t$  is *essential* if both components of the complement of  $\tilde{\mathcal{C}} - \tilde{t}$ , a lift of  $t$ , contain points arbitrarily far from  $\tilde{t}$ .

We have the following characterization of such tracks:

**Lemma 2.9.** *Let  $t \subset \mathcal{C}$  be a non-essential track. Then the splitting of  $G$  given in Proposition 2.4. Will be of the form*

$$G = A *_C C$$

*Proof.* If  $G$  splits as an HNN, the track  $t$  cannot be non-essential. Otherwise  $G$  splits as an amalgam

$$G = A *_B C$$

w.l.o.g. we may assume  $A \supsetneq B \Rightarrow [A : B] \geq 1$ . Considering the Bass-Serre  $T$  as a topological quotient of the universal cover gives us that  $t$  is nonessential if and only if  $[C : B] = 1$ .  $\square$

Topologically, we have:

**Lemma 2.10** (Lemma 2.3 of [DS99]). *Suppose  $P$  is a pattern of  $\tilde{\mathcal{C}}$  so that the quotient in  $\mathcal{C}$  represents a splitting and has a minimal number of components. Then every track in  $P$  is essential.*

We now have this critical fact:

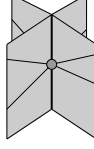


Figure 1: An edge midpoint and its book neighbourhood, with train tracks

**Theorem 2.11.** *A group  $G = \pi_1(\mathcal{C}, x_0)$  is freely decomposable if and only if there exists an essential track  $t \subset \mathcal{C}$  such that  $Gp(t) = 1$ .*

*Proof.* Suppose such a track existed then Proposition 2.4 gives the desired splitting. Conversely, if  $G$  is freely decomposable, then it acts on its Bass-Serre tree  $T$  and Proposition 2.6, gives us a resolution of this action that contains a track with the desired property.  $\square$

### 2.3 $\pi_1$ -train tracks

A pattern in a cell complex is the generalization of a multicurve in a surface. We therefore borrow some machinery. We introduce measured  $\pi_1$ -train tracks, which are weighted subgraphs of  $\mathcal{C}$ , that in a sense enable us to “compress” a pattern  $P$ .

**Definition 2.12.** For the midpoint  $x$  of each 1-cell  $e$  in  $\mathcal{C}$ , there is an open neighbourhood  $B_x$  that is homeomorphic to the identification space

$$(\sqcup_{i=1}^{n_x} P_i) / \sim$$

where each  $P_i$  is a copy of  $[0, 1) \times (0, 1)$  and  $\sim$  corresponds to gluing the  $P_i$  together along the line  $\{0\} \times (0, 1)$ .  $B_x$  is called the *book neighbourhood* of  $x$  and the interiors of the  $P_i \subset B_x$  are called *pages*. (See Figure 1.)

**Definition 2.13.** A  $\pi_1$ -train track  $t$  is a graph embedded in  $\mathcal{C}$  such that:

- Each vertex of  $t$  coincides with the midpoint of a 1-cell of  $\mathcal{C}$ .
- For each 1-cell midpoint  $x$  that coincides with a vertex of  $t$ , for each page  $P_i$  of  $B_x$ ,  $t \cap P_i \neq \emptyset$ .

**Definition 2.14.** An  $\mathbb{N}$ -measured  $\pi_1$ -train track is a train track  $t$  such that each edge  $e$  has a measure  $m(e) \in \mathbb{N}$ . Moreover these measures must satisfy the *page condition*: for each 1-cell midpoint  $x$ , for any two pages  $P_i, P_j$  of  $B_x$  we must have an equality:

$$\sum_{\alpha \in A_j} m(e_\alpha) = \sum_{\beta \in A_i} m(e_\beta)$$

where  $\{e_\alpha \mid \alpha \in A_i\}$  and  $\{e_\beta \mid \beta \in A_j\}$  are the edges of  $t$  that intersect  $P_i$  and  $P_j$  respectively.

The terminology is motivated by the fact that this construction gives exactly the  $\pi_1$ -train tracks defined in [BS88]. We moreover wish to point out that the page condition is an analogue of the *switch condition* that is appropriate when  $\mathcal{C}$  is not a surface. We finally note that this definition

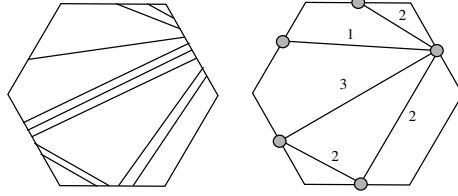


Figure 2: Going from  $\pi_1$ -train tracks to patterns

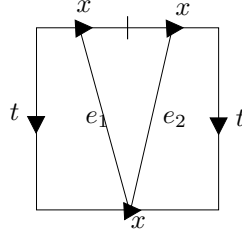


Figure 3: A train track in  $BS(1, 2)$

includes the possibility that some book neighbourhood of some  $x$  has only one page, in which case the page condition vacuously holds at  $x$ . We now have “Dehn coordinates” for patterns:

**Proposition 2.15.** *Each pattern  $P \subset \mathcal{C}$  (not necessarily two-sided) on  $\mathcal{C}$  gives rise to a  $\mathbb{N}$ -measured  $\pi_1$ -train track  $t \subset \mathcal{C}$ , conversely every  $\mathbb{N}$ -measured  $\pi_1$ -train track  $t \subset \mathcal{C}$  gives a pattern  $P \subset \mathcal{C}$ .*

*Proof.* The definition of a pattern, implies that is embedded in some sort of “normal position”. Consider the correspondence given in Figure 2. The definition of a pattern implies that the page condition must be satisfied, the converse must hold as well. We see that a  $\pi_1$ -train track defines a pattern that is unique up to isotopy of  $\mathcal{C}$  that  $\mathcal{C}^{(1)}$  invariant.  $\square$

**Definition 2.16.** An unmeasured  $\pi_1$ -train track is said to be *formally consistent* if we can assign measures, which are not all zero, to the edges that satisfy the page condition.

We now give an example of a formally inconsistent train track.

**Example 2.17.** Let  $\mathcal{C}$  be the cell complex for the group  $BS(1, 2) = \langle x, t \mid t^{-1}xt = x^2 \rangle$ . Consider the following  $\pi_1$ -train track given in Figure 3.

By the page condition we have the equality

$$m(e_1) + m(e_2) = m(e_1) = m(e_2)$$

which only has the zero solution over  $\mathbb{R}$ . This is expected because two conjugate elements must have the same translation length, so in particular a hyperbolic element cannot be conjugate to its square.

What is great about  $\pi_1$ -train tracks is that they turn the seemingly intractable space of patterns (modulo isotopies of the ambient cell complex which fix the 0-skeleton and map the 1-skeleton into itself), which in turn encodes essential information about all actions of our group on simplicial trees, into a finite union of linear subsets of  $\mathbb{N}^d$  where  $d$  is easily computed from the  $\pi_1$ -train track.

## 2.4 Band Complexes

Band complexes are 2-complexes with added structure. Our definition will very much resemble the construction given in [BF95]. For purposes, however, we shall be considering *unmeasured band complexes* which because of all the combinatorial data we want to keep track of will have some added features. Following [BF95] we have.

**Definition 2.18.** A *band*  $B$  is a product  $J_B \times [0, 1]$  where  $J_B \subset \mathbb{R}$  is a closed interval. We call the subspaces  $J_B \times \{0\}$  and  $J_B \times \{1\}$  *bases*. For some base  $\mu = J_B \times \{s\}$ , where  $s = 0, 1$ , we denote by  $\bar{\mu}$  the base  $J_B \times \{1 - s\}$ .  $\bar{\mu}$  is called the *dual of*  $\mu$ . A *vertical subset of a band* is a subset of the form  $\{z\} \times [0, 1]$ . For a base  $\mu$  we will denote the band that contains it  $B_\mu$ .

We will usually denote bases by Greek letters  $\mu$  and  $\lambda$ .

**Definition 2.19.** A *union of bands*  $Y$  consists of a graph (i.e. a 1-complex)  $\Gamma$  with bands  $B_1, \dots, B_n$  attached to  $\Gamma$  via embeddings

$$\mu \hookrightarrow \text{int}(\Gamma - \Gamma^0)$$

i.e. bases are sent to the interiors of edges.

**Definition 2.20.** A *band complex*  $\mathcal{C}$  is a 2-complex obtained by taking a band complex  $Y$ , and attaching 2-cells  $D_1, \dots, D_m$  via immersions

$$\iota_i : \partial D_i \rightarrow Y$$

such that for each band  $B \subset Y$  we have  $\iota_i(\partial D_i) \cap B$  is a *disjoint union of vertical subsets*

**Convention 2.21.** Although a band complex is just a normal 2-complex: the bands are essentially 2-cells, we will always distinguish between “normal” 2-cells and bands. In particular a band will *never* be referred to as a 2-cell.

The following are definitions motivated by *generalized equations*. Generalized equations were constructed by Makanin in [Mak82] and were the main objects used in his proof of decidability of the existential theory of free groups. We refer the reader, for example, to Section 4 of [KM05a] for definitions of generalized equation.

**Definition 2.22.** Let  $\mathcal{C}$  be a band complex with union of bands  $Y \subset \mathcal{C}$ . Let  $\Gamma \subset Y$  be as in Definition 2.19. We denote

$$I(\mathcal{C}) = \left( \bigcup_{\text{bases } \mu} \mu \right) \subset \Gamma$$

In general  $I(\mathcal{C})$  is not connected, we can embed  $I(\mathcal{C})$  into  $\mathbb{R}$  via a continuous map  $i : I(\mathcal{C}) \rightarrow \mathbb{R}$  such that the number of connected components of the closure of  $i(I(\mathcal{C}))$  is the same as the number of connected components of  $I(\mathcal{C})$ . We denote the convex hull of the image  $(I(\mathcal{C}), i)$  and call it the *interval associated to the band complex  $\mathcal{C}$* .

When denoting such intervals, we will usually not express  $i$  explicitly. As observed in [BF95, GLP94] the bands define partial homeomorphisms of  $\mathbb{R}$ . Each band  $B$  induces two maps: one sending  $i(int(\mu))$  to  $i(int(\bar{\mu}))$  and its inverse. We denote this map  $f_\mu : i(int(\mu)) \rightarrow i(int(\bar{\mu}))$  and we have

$$f_\mu(i((z, s))) = i((z, 1 - s))$$

where  $s = 0, 1$  and  $B = \{(z, t) \mid z \in J_B, t \in [0, 1]\}$ . We moreover point out that  $f_\mu$  extends to a homeomorphism  $\hat{f}_\mu : \mathbb{R} \rightarrow \mathbb{R}$ . For each base  $\mu$  let  $b_1, b_2$  be its endpoints. We denote

$$\alpha(\mu) = \inf\{i(x) \mid x \in int(\mu)\}, \beta(\mu) = \sup\{i(x) \mid x \in int(\mu)\}$$

The reason for this rather contrived definition is because two distinct closed sections (defined later) may have common endpoints.

**Definition 2.23.** We say that a band  $B$  or a pair of bases  $(\mu, \bar{\mu})$  is *orientation reversing* if the homeomorphism  $\hat{f}_\mu : \mathbb{R} \rightarrow \mathbb{R}$  is orientation reversing.

The terminology of the next two definitions comes from their analogues in generalized equations.

**Definition 2.24.** Let  $\iota_i : \partial D_i \rightarrow Y$  be a 2-cell attaching map. We have that  $\iota_i(\partial D_i)$  contains a segments  $c_1, \dots, c_k(i)$  consisting of maximal connected subsets of

$$\bigsqcup_{\text{bands } B} (B \cap \iota_i(\partial D_i))$$

that lie inside some band. We call these  $c_i$  *2-cell induced boundary connections*, we moreover have a relation  $x \sim_{c_i} y$  where  $x, y \in I(\mathcal{C})$  are the endpoints of  $c_i$ . More generally, a *boundary connection* will be a marked maximal vertical subset  $S \subset B$ , inducing a relation  $x \sim_S y$ , where  $x, y$  are endpoints of  $S$ .

**Convention 2.25.** We will also consider the edges of bands to be boundary connections.

**Definition 2.26.** We call the set of points  $\{x_1, \dots, x_{m+1}\} \subset I(\mathcal{C})$  corresponding endpoints of bases  $\mu$  as well as endpoints of boundary connections *boundaries*. We will call the subintervals  $h_1, \dots, h_m \subset I(\mathcal{C})$  corresponding to the closures of the connected components of  $I(\mathcal{C}) - \{x_1, \dots, x_{m+1}\}$  *items*.

As a convention the subscripts for boundaries and items are ordered w.r.t. the induced order on  $I(\mathcal{C})$ , so that we have

$$h_i = [x_i, x_{i+1}]$$

We call a band complex  $Y$  with the additional structure of having boundary connections and items an *unmeasured band complex*, although we will usually simply call it a *band complex*.



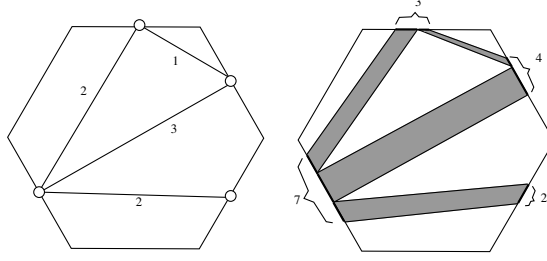


Figure 4: How to get a measured band complex from a  $\pi_1$ -train track

**Definition 2.27.** An  $\mathbb{N}$ -*measured* band complex is a band complex equipped with a “measure function”  $|\cdot|_m : \{h_1, \dots, h_r\} \rightarrow \mathbb{N}$  on its items, where  $\mathbb{N}$  is an abelian group. This function extends to intervals  $[x_i, x_j] \leq I(\mathcal{C})$  where  $i < j$  by setting  $|[x_i, x_j]|_m = |h_i|_m + \dots + |h_{j-1}|_m$ . We moreover require the following linear equations to hold:

- (a)  $|[x_i, x_j]|_m = |[x_k, x_l]|_m$  where the intervals  $[x_i, x_j], [x_k, x_l]$  correspond to the images of bases  $\mu, \overline{\mu}$  respectively.
- (b)  $|[x_i, x_j]|_m = |[x_k, x_l]|_m$ , where  $x_i \neq x_j$  and  $x_k \neq x_l$  and there are boundary connections connecting  $x_i, x_j$  to  $x_k, x_l$  either respectively or inversely.

**Definition 2.28.** An unmeasured band complex  $\mathcal{C}$  is said to be *formally consistent* if it satisfies the following:

- (a) If  $(\mu, \overline{\mu})$  is orientation reversing then  $\mu \cap \overline{\mu} = \emptyset$ .
- (b)  $\mathcal{C}$  admits an  $\mathbb{N}$ -measure.

### 2.4.1 Correspondences

Looking at Figure 4 we see that given a track  $t \subset \mathcal{C}$ , then  $\mathcal{C}$  can be realized as a corresponding  $\mathbb{Z}$ -measured band complex. Moreover Figure 5 gives us a convention on how to embed a pattern into a  $\mathbb{Z}$  measured band complex (it is easy to see that the embedded object satisfies the axioms of a pattern).

**Definition 2.29.** We call the pattern given above the *standard embedded pattern* or a *measure*  $m$ . For a measure  $m$  on  $\mathcal{C}$  we denote the standard embedded pattern  $P_m$

## 3 The elimination process on band complexes

The elimination process is a term coined by Kharlampovich and Myasnikov for their modification of Razborov’s algorithm, in analogy to elimination theory in commutative algebra.

Although what we will do here more closely resembles Makanin’s algorithm in that we simply wish to establish whether there exists an essential

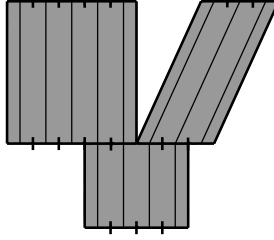


Figure 5: The standard embedded pattern in a  $\mathbb{Z}$ -measured band complex

$\pi_1$ -trivial track, the term still seems appropriate in that we will in a sense be considering all possible combinatorial possibilities and discarding inappropriate situations.

The elimination process, takes an unmeasured band complex and outputs an arborescence  $T(\mathcal{C})$  of band complexes (see Section 3.3.) We are searching for free splittings of  $G$ , which means that we are searching for patterns with  $\pi_1$ -trivial tracks. Our main result, Theorem 1.1 follows from the fact that the leaves of  $T(\mathcal{C})$  are band complexes  $\mathcal{C}_i$  such that there are no remaining bands but from which we can easily read off a free decomposition. We will therefore prove the following theorem:

**Theorem 3.1.** *Let  $G = \langle X \mid R \rangle = \pi_1(\mathcal{C}, x_0)$  and suppose there is a known solution to the word problem for  $\langle X \mid R \rangle$ . Then it is possible to effectively construct a finite tree  $T_0(\mathcal{C}) \subset T(\mathcal{C})$  with the property that if there exists an essential track  $t \subset \mathcal{C}$  with  $Gp(t) = \{1\}$ , then there is a branch in  $T_0(\mathcal{C})$  that enables us to effectively construct an essential track  $t' \subset \mathcal{C}$  such that  $Gp(t') = \{1\}$ .*

In particular, if  $G$  is not freely decomposable, then this will be visible from the finite and algorithmically constructible  $T_0(\mathcal{C})$ .

### 3.1 Complexities of Band complexes

**Definition 3.2.** Let  $D$  be a 2-cell of  $\mathcal{C}$  attached via the immersion

$$\iota : \partial D \rightarrow \mathcal{C}$$

there are  $k(D)$  segments  $c_1, \dots, c_{k(D)}$  given as in Definition 2.24. We call the segments  $s_1, \dots, s_{k(D)} \subset \partial D$  that map onto the  $c_i$  *vertical sections*. We call  $k(D)$  the *vertical length* of  $D$ .

**Definition 3.3.** Let

$$S = \bigcup_{\text{bases } \mu} \text{int}(\mu)$$

and let  $S_\alpha; \alpha \in A$  be the connected components of  $S$ . Let  $\sigma_\alpha$  be the closure of  $S_\alpha$  then  $\sigma_\alpha$  is called a *section*. We note that  $I(\mathcal{C})$  is the union of its sections. A connected union of items is called a *closed section*

**Definition 3.4.** For a section  $\sigma \subset I(\mathcal{C})$  we define the *complexity* of  $\sigma$ :

$$n(\sigma) = \max((\text{the number of bases inside } \sigma) - 2, 0)$$

and for  $\mathcal{C}$  we define the *complexity* of  $\mathcal{C}$ :

$$\tau(\mathcal{C}) = \sum_{\sigma\text{-sections}} n(\sigma)$$

For some union of section  $J \subset I(\mathcal{C})$  we denote

$$\tau(J) = \sum_{\sigma \subset J} n(\sigma)$$

For an item  $h \in I(\mathcal{C})$  we define:

$$\gamma(h) = \{\mu \in \text{Bases} \mid \mu \supset h\}$$

finally we define the *total complexity* of a band complex to be the sum of the number of items, the number of bases, the number of boundary connections, and the vertical lengths of all the 2-cells.

### 3.2 Definitions of the moves: where the fun begins...

We refer the reader to [BF95] for pictures and insight. We will be following [KM05a] Section 5 for the definitions. Recall that we are looking at unmeasured band complexes in the hope of finding a  $\pi_1$ -trivial track. *All of these moves preserve the fundamental group of  $\mathcal{C}$ .*

**Definition 3.5.** A 2-cell  $D \subset \mathcal{C}$  has a *free face* if a point on the image of its boundary has a neighbourhood homeomorphic to  $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ . A 2-cell is a *balloon* if its boundary is mapped to a point.

This lemma is obvious:

**Lemma 3.6.** *If a 2-cell  $D \subset \mathcal{C}$  has a free face or is a balloon, then it can be removed from  $\mathcal{C}$  without changing the fundamental group.*

**Definition 3.7.** A 2-cell  $D$  is said to be collapsable if the attaching map  $\iota : \partial D \rightarrow \mathcal{C}$  is a homeomorphism.

This lemma will be useful for dealing with bigons.

**Lemma 3.8.** *If the boundary of a collapsable cell is the union of two arcs  $\alpha, \beta$  then we can change  $\mathcal{C}$  by collapsing  $D$  so that  $\alpha$  and  $\beta$  are identified, and the fundamental group is invariant.*

**Definition 3.9.** We always assume that the band complex  $\mathcal{C}$  is formally consistent. We have the following five *elementary transformations*:

- **ET1: Cutting a base:** Let  $c_i \subset B$  be a boundary connection in a band. We cut  $B$  along  $c_i$  and glue in a disc (a bigon.) The resulting band complex has two more bases.
- **ET2: Transfer a base:** Let  $\mu, \lambda$  be bases with  $\mu \supset \lambda$ , suppose moreover that for each boundary in  $x_i$  in  $\lambda$  there is a boundary connection  $c_i$  in  $B_\mu$  connecting  $x_i$  to some  $z_i \in \overline{\mu}$ . Then we *slide*  $\lambda$  inside  $\mu$  along vertical subsets of  $B_\mu$  so that  $\lambda$  now sits in  $\overline{\mu}$ , moreover 2-cells attaching maps get “stretched” as in Figure 6.

- **ET3: Remove a pair of matched bases:** Suppose we have the bases  $\mu, \bar{\mu}$  coincide. Then the band  $B_\mu$  is an annulus. By taking a loop in  $B_\mu$  we can check if  $\text{Gp}(B_\mu) \leq \mathbb{Z}$  is trivial. If so we fill in this annulus with a cylinder  $D^2 \times \mu$  and collapse the cylinder onto  $\mu$ . This will change 2-cell attaching maps that went through  $B_\mu$ . We remove any balloons that might be formed. The result is a band complex with fewer bases. If  $\text{Gp}(B_\mu) \neq \{1\}$  then we say that  $\mathcal{C}$  is *unsuitable*.
- **ET4: Remove a lonely base:** Suppose that some base  $\mu$ 's interior doesn't intersect any other bases, then we collapse  $B_\mu$  onto  $\bar{\mu}$ , we modify 2-cell attaching maps as follows: if a segment of a 2-cell attaching map goes through the interior of  $B_\mu$  then we collapse that segment to a point; otherwise we collapse the interior of  $B_\mu$  but do not touch the 2-cell, if this gives a 2-cell with a free face we remove it. If the 2-cell is collapsible we collapse the arc that used to lie in  $B_\mu$  onto the other half and this modifies other 2-cell attaching maps. *We also call this move collapsing  $B_\mu$  onto  $\bar{\mu}$ .*
- **ET5: Introduce a boundary connection:** Let  $x \in \mu$  and  $z \in \bar{\mu}$  be points such that there is a line in  $B_\mu$  that doesn't intersect any boundary connections in  $B_\mu$  then we reparametrize  $J_B \times [0, 1] \rightarrow B_\mu$  so that all previous boundary connections are vertical subsets and so that  $x$  and  $z$  now lie in the same vertical subset. We now make the boundary connection in  $B_\mu$  between  $x$  and  $z$ . If  $x \in \mu$  and we created a boundary connection in  $B_\mu$  that contained  $x$  we will say that *we  $\mu$ -tied  $x$*
- **ET6: Add a hanging band:** Create a new band  $B_\mu$  and attach  $\mu$  inside  $I(\mathcal{C})$ , the new interval will be  $I(\mathcal{C}) \cup \bar{\mu}$ .
- **ET7: Merge bands:** Suppose we have bases  $\mu, \lambda$  that coincide and suppose moreover that the interior of  $\mu$  doesn't touch the interior of any other bands. Then we transfer  $\lambda$  through  $B_\mu$  (ET2) and remove the lonely base  $\mu$  (ET4).
- **ET8: Periodic Merger:** See Definition 4.25. For now all that we need to know is that it decreases the  $\tau$ -complexity by 2.

If the band complex is  $\mathbb{R}$ -measured, then everything is rigid and there is only one possibility for move ET5. However, for a combinatorial band complex, ET5 can be seen as adding new variables or equations to the system of equations given in Definition 2.27. *We need this flexibility to be able to enumerate all possible patterns in  $\mathcal{C}$ .*

Although this fact should be more or less obvious, it seems worthwhile to state it precisely.

**Proposition 3.10.** *Let  $\mathcal{C}'$  be a measured band complex obtained from  $\mathcal{C}$  after applying an elementary transformation  $\mathcal{C} \rightarrow \mathcal{C}'$ . Then  $\pi_1(\mathcal{C}') \approx \pi_1(\mathcal{C})$  and if  $P' \subset \mathcal{C}'$ ,  $P \subset \mathcal{C}$  are the standard embedded patterns (see Figure 5) then  $\text{Gp}(P) \approx \text{Gp}(P')$ .*

*sketch.* This is clearly the case for ET1, ET3, ET5, ET6. The main problem in defining the homomorphism between the fundamental groups of the band complexes lies in the fact that there is no obvious continuous

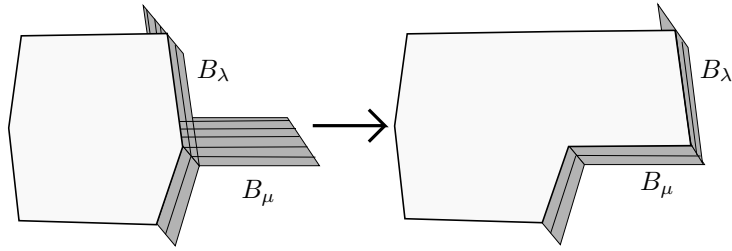


Figure 6: How 2-cell attaching maps get modified via ET2

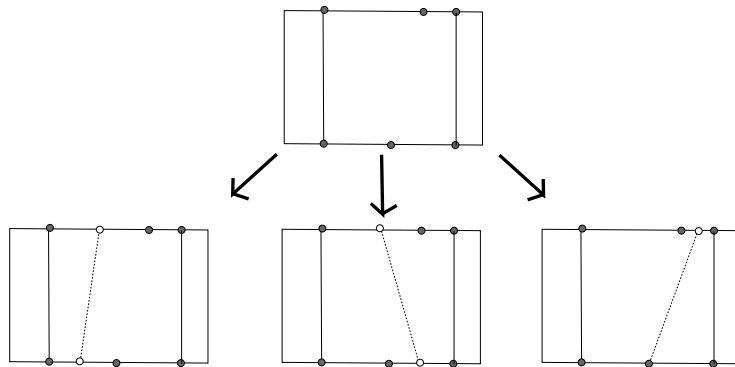


Figure 7: Three combinatorially different possibilities for ET5. The grey points are existing boundaries, the lines are boundary connections and the dotted lines represent the boundary connections we wish to create. Note that we can always reparametrize so that the dotted lines are vertical

map between the two spaces. Let  $\gamma$  be some based loop in  $(\mathcal{C}, x_0)$  then, replacing  $\gamma$  by a homotopic loop with same basepoint if necessary, we can arrange so that  $\gamma$  only intersects bands in their bases or in vertical subsets and does not intersect the interior of the 2-cells. We say such a loop is in *regular position*. In which case, it is easy to see how we can “stretch” or “collapse” segments in  $\gamma$  to get some loop  $\gamma'$  in  $(\mathcal{C}', x'_0)$  (see Figure 6.) Call this map  $d$ .

Similarly we can reverse the move and get a map from based loops in regular position lying in  $\mathcal{C}'$ , call this map  $d'$ . Although  $d$  and  $d'$  are not inverses checking ET2, ET4, and ET7 it is easy to see that the curve  $\gamma \subset \mathcal{C}$  is homotopic to  $d' \circ d(\gamma)$  and similarly  $\gamma' \subset \mathcal{C}'$  is homotopic to  $d \circ d'(\gamma')$ . The advertised isomorphisms therefore follow.  $\square$

**Lemma 3.11.** *If we have band complexes*

$$\mathcal{C}_1 \xrightarrow{ET_i} \mathcal{C}_2$$

*such that  $\mathcal{C}_2$  is obtained from  $\mathcal{C}_1$  by an elementary transformation then any measure on  $\mathcal{C}_2$  induces a measure on  $\mathcal{C}_1$ . Specifically the measure of each item in  $\mathcal{C}_1$  is given by a fixed positive linear combination of the measures of the items in  $\mathcal{C}_2$ . In particular any measure on  $\mathcal{C}_2$  induces a measure on  $\mathcal{C}_1$ .*

*Proof.* This follows by direct inspection of the elementary transformations ET1 - ET7. When we present ET8 (see Lemma 4.23) we will check this.  $\square$

### 3.3 The Process

We now describe the process on band complexes, there are two main subprocesses *thinning* and *induction* (these are Process 1 and Process 2 of [BF95] respectively.) These processes are repetitions of two *composite* moves, which we now present:

#### 3.3.1 There is an item $h$ such that $\gamma(h) = 1$ : the thinning move

In this case we do a *thinning move*. First some definitions.

**Definition 3.12.** Let  $\mu$  be a base that contains an item  $h \subset \mu$  such that  $\gamma(\mu) = 1$  has a *free segment*. We say that  $\mu$  is:

- free if its interior doesn't meet any other bases.
- half-naked if  $\mu$  has an endpoint  $z$  such that there is an open neighbourhood  $N(z) \subset \mu$  such that  $N(z) - \{z\}$  doesn't meet any other base. A maximal such  $N(z)$  is called a *collapsable segment*.
- splittable if it is neither free nor half naked. In such a case there is some  $z \in \mu$  and a maximal w.r.t. inclusion open neighbourhood  $N(z) \subset \mu$  that doesn't meet any other base, which we also call a *collapsable segment*.

The thinning move will first consist of one of these possible collapses:

1. If there is a free base  $\mu$ , then it is lonely and we apply ET4 to  $\mu$ , i.e. we collapse the band  $B_\mu$  onto  $\overline{\mu}$ .
2. If there is a half naked base  $\mu$  but no free bases, then we  $\mu$ -tie (ET5) the endpoint of a collapsable segment, giving a boundary connection  $c$ . We cut  $B_\mu$   $c$  (ET1), giving bands  $B_{\mu'}$  and  $B_{\mu''}$ , one of which, say  $B_{\mu'}$  is free. We now collapse  $B_\mu$ .
3. If there are no half naked bases, nor free bases, then we pick some splittable base  $\mu$  and take a collapsable segment  $N(z) \subset \mu$ . We  $\mu$ -tie the endpoints of  $N(z)$  (ET5) and cut  $B_\mu$  into three bands (ET1 twice) and collapse the band containing the new free base.
4. If we have matched bases we either remove them (ET4), or we declare  $\mathcal{C}$  to be *unsuitable* and we stop.

And then if there is a resulting section  $\sigma$  with  $n(\sigma) = 2$  and  $\sigma = \lambda_1 = \lambda_2$  then we merge the bands  $B_{\lambda_1}$  and  $B_{\lambda_2}$ .

**Lemma 3.13.** *A thinning move doesn't increase the complexity  $\tau(\mathcal{C})$ .*

*Proof.* Collapsing a free base decrease the number of bands and deletes a section, so the  $\tau$  complexity decreases. Collapsing a collapsable segment of a half naked base, doesn't change the number of more bases or the number of sections, and may increase the number of sections.

If there are only splittable bases, then in particular no splittable base  $\mu$  lies in a section  $\sigma$  s.t.  $n(\sigma) \geq 3$ . Now collapsing the collapsable segment of  $\mu$  gives us at least two sections  $\sigma', \sigma'' \subset \sigma$ , if there are more than two new sections we glue sections together and our estimate will give an upper bound.

Since the total number of bands increases by 2 and the number of sections increases by (at least) 1 and because  $n(\sigma'), n(\sigma'') > 0$ , this is like adding 2 and subtracting 2 from  $\tau(\mathcal{C})$ , so the  $\tau$  complexity doesn't increase.  $\square$

### 3.3.2 For each item $h$ we have $\gamma(h) \geq 2$ : the entire transformation

**Definition 3.14.** We have an order  $<$  on  $I(\mathcal{C})$  induced by  $i : (\mathcal{C}) \rightarrow \mathbb{R}$ . A *leading base* is a base  $\mu$  such that  $\alpha(\mu)$  is minimal. A *carrier base* is a longest leading base. If  $\mu$  is a carrier base then any other leading base  $\lambda$  is a *transfer base w.r.t.*  $\mu$ .

The *entire transformation* is the following sequence of elementary transformations:

1. Pick a carrier base  $\mu$ .
2. If applicable apply a periodic merger (ET8).
3. For every transfer base  $\lambda$ , we  $\mu$ -tie all the boundaries in  $\lambda$  (ET5) and then we transfer  $\lambda$  to  $\overline{\mu}$  (ET2).
4. Once all the transfer bases have been transferred, we can cut  $\mu$  into two bases  $\mu', \mu''$  (ET1) such that  $\mu'$  is the maximal initial segment of  $\mu$  whose interior does not meet any other bases.  $\mu'$  is now lonely so we remove it (ET4).

5. If we have matched bases we either remove them (ET4), or we declare  $\mathcal{C}$  to be *unsuitable* and we stop.

We invite the reader to check this fact directly, or they can check [BF95] section 7 or [KM05a] Lemma 18:

**Lemma 3.15.** *Entire transformations don't increase the  $\tau$ -complexity.*

We can now define the two main subprocesses of the elimination process.

**Definition 3.16.** A *thinning process* is a sequence of thinning moves  $\mathcal{C}_v \rightarrow \mathcal{C}_{v_1} \rightarrow \dots$  applied to band complexes. An *induction process* is a sequence of entire transformations  $\mathcal{C}_v \rightarrow \mathcal{C}_{v_1} \rightarrow \dots$  applied to band complex

The term induction comes from the technique used in the study of interval exchange maps, see [Rau79, Vee82].

### 3.4 $T(\mathcal{C})$ and reconstructing the pattern

Having defined these two moves, we are in a position to define the directed rooted tree  $T(\mathcal{C})$ . To each the vertex of  $T(\mathcal{C})$ , except the root, we associate a band complexes:

- To the root we assign the cell complex  $\mathcal{C}$ .
- We new enumerate the finite collection  $t_1, \dots, t_n$  of formally consistent  $\pi_1$ -train tracks, the children of  $\mathcal{C}$  will be assigned to the corresponding unmeasured band complexes.
- We construct the rest of the tree inductively by associating to each child of a  $v$  corresponding to a band complex  $\mathcal{C}_v$  one of the possible unmeasured band complexes that can be obtained from  $\mathcal{C}_v$  by either the applicable thinning move or entire transformation.
- Unsuitable band complexes have no children.
- We will also have *leaves* corresponding to band complexes without any bases left (they all got matched with their duals).

Now we note that if we put a  $\mathbb{N}$ -measure on a band complex (which could also be interpreted as an  $\mathbb{R}$ -measure), whenever we apply (ET5) or (ET8) there is only one possibility. Moreover we note that  $I(\mathcal{C})$  now has a definite size, i.e. its Lebesgue measure, and applying entire transformations or thinning moves decreases the measure of  $I(\mathcal{C})$ . It follows that  $T(\mathcal{C})$  is just a branch, and if  $\mathcal{C}$  in fact had a  $\mathbb{N}$ -measure then the process terminates.

If we suitably modify ET3, consider arbitrary  $\mathbb{R}$  measures and relax our consistency requirements we recover the celebrated Rips machine or, if we have a  $\mathbb{N}$ -measure, we get “an orbit counting algorithm” as in [AHT06]. Given the relationship between patterns and  $\mathbb{N}$ -measured band complexes we have the following lemma:

**Lemma 3.17.** *A pattern  $P$  such that  $Gp(P) = \{1\}$  induces a finite sequence of moves on a band complex such that any base is either eventually matched with its dual or the band that contains it is collapsed to an interval. This sequence of moves corresponds to a branch in  $T(\mathcal{C})$  from the*



root to a leaf. Conversely any such sequence of band complexes ending in a suitable leaf enables one to produce a pattern  $P \subset \mathcal{C}$  with  $Gp(P) = \{1\}$ .

*Proof.* The  $(\Rightarrow)$  part follows immediately from the previous discussion. So suppose we have a branch in  $T(\mathcal{C})$  of the form

$$\mathcal{C} \rightarrow \mathcal{C}_{v_1} \rightarrow \dots \rightarrow \mathcal{C}_{v_m}$$

where  $\mathcal{C}_{v_m}$  is a leaf, i.e.  $\mathcal{C}_{v_m}$  has no more bases. Then by definition of ET3, we have that

$$\mathcal{C}_{v_m} = \mathcal{C}' \cup \sigma \cup \mathcal{C}'' \text{ or } \mathcal{C}' \cup \sigma$$

where  $\mathcal{C}', \mathcal{C}''$  are cell complexes and  $\sigma$  is circle or line segment that intersects  $\mathcal{C}' \cup \mathcal{C}''$  in one or two points respectively.  $\sigma$  may still be divided into  $n$  items. We give to one of these items measure 1, and the others measure 0. Reversing ET3 or ET4, gives us a pair of with measure 1, moreover the standard embedded pattern  $P_{m-1} \subset \mathcal{C}_{v_{m-1}}$  will satisfy  $Gp(P_{m-1}) = \{1\}$ . The critical observation is that if we have a transformation  $\mathcal{C} \rightarrow \mathcal{C}'$  and  $\mathcal{C}'$  is measured, then there is a unique compatible measure on  $\mathcal{C}$ : the measures of the items in  $\mathcal{C}$  are given by specific linear combinations of the measures of the items of  $\mathcal{C}'$ . Reversing all the moves thus gives us a sequence of measured band complexes which ultimately enables us, via standard embedding, to construct a pattern in  $P \subset \mathcal{C}$ . Moreover by Proposition 3.10  $Gp(P) = \{1\}$ .  $\square$

It therefore follows that if  $\mathcal{C}$  has a  $\pi_1$  trivial track, then it will correspond to a leaf of the possibly infinite tree  $T(\mathcal{C})$ .

### 3.5 Infinite branches of $T(\mathcal{C})$ and the finite subtree $T_0(\mathcal{C})$

**Definition 3.18.** Let  $P \subset \mathcal{C}$  be a pattern. The *length*  $|P|$  of the pattern is the sum of the measures of the branches of the associated  $\pi_1$ -train track. A pattern  $P$  such that  $Gp(P) = \{1\}$  is said to be *1-minimal* if it has minimal length among all other patterns  $P'$  such that  $Gp(P') = \{1\}$ . Similarly we say a measure  $m$  on a band complex is 1-minimal if the sum of the measures of the items is minimal such that the canonical embedded pattern gives a free splitting.

**Proposition 3.19.** Let  $m$  be a 1-minimal measure on  $\mathcal{C}_1$  and suppose we have that  $\mathcal{C}_2$  is obtained from  $\mathcal{C}_1$  from an elementary transformation:

$$\mathcal{C}_1 \xrightarrow{ET_i} \mathcal{C}_2$$

then the induced measure  $m'$  on  $\mathcal{C}_2$  is also 1-minimal.

*Proof.* Follows immediately from Lemma 3.11.  $\square$

**Definition 3.20.** A path

$$p_i : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \dots \rightarrow \mathcal{C}_i$$

in  $T(\mathcal{C})$  is *unsuitable* if it cannot be the initial segment of a path from  $\mathcal{C}$  to a leaf corresponding to a 1-minimal measure.

**Definition 3.21.** The tree  $T_0(\mathcal{C})$  is the minimal subtree of  $T(\mathcal{C})$  containing all paths from  $\mathcal{C}$  to leaves which correspond to 1-minimal patterns. If there are none then  $T_0(\mathcal{C})$  is empty.

Our goal is now to show that we can in fact algorithmically construct this tree.

**Definition 3.22.** A band complex with an item  $h$  such that  $\gamma(h) = 1$  is called *thinnable*. A segment  $J$  (see Definition 3.3) such that for each item  $h \in J$  we have  $\gamma(h) = 2$  is called a *quadratic segment*. A generalized equation such that for each item we have  $\gamma(h) \geq 2$  is called *unthinnable*.

We chose the term quadratic is due its historical importance with respect quadratic equations over free groups or monoids. We first have a little lemma.

**Lemma 3.23.** Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a (possibly finite) path in  $T(\mathcal{C})$ . This sequence can be seen as being produced by an alternating sequence of thinning and induction processes. The number of times we can switch is bounded by the complexity  $\tau(\mathcal{C})$ .

*Proof.* Neither thinning nor entire transformation increase the complexity  $\tau$ , suppose we are doing an induction process, this means that for each item  $h$ ,  $\gamma(h) \geq 2$ , i.e. the band complex is unthinnable. Straightforward checking shows that if an entire transformation doesn't decrease the complexity, then the resulting band complex is still unthinnable. The result now follows.  $\square$

**Definition 3.24.** Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a (possibly finite) path in  $T(\mathcal{C})$  obtained by an induction process. Suppose that each  $I(\mathcal{C}_i)$  is quadratic path  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  is called *quadratic*. Otherwise the path is called *axial*.

**Definition 3.25.** Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a (possibly finite) path in  $T(\mathcal{C})$  obtained by a thinning process. Then the path is said to be *thinning*.

**Lemma 3.26.** Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be an infinite branch in  $T(\mathcal{C})$  then it has a tail  $\mathcal{C}_N \rightarrow \mathcal{C}_{N+1} \rightarrow \dots$  that is either:

1. *thinning*;
2. *quadratic*; or
3. *axial*.

*Proof.* By Lemma 3.23 we eventually only either do a thinning process or an induction process, the possibilities given in the latter case are mutually exclusive so the result holds.  $\square$

**Corollary 3.27.** If we have a computable upper bound  $f$ , depending on the band complexes, for the lengths of the thinning, quadratic, and axial subpaths that make up a branch in  $T(\mathcal{C})$  from the root to a leaf that gives a 1-minimal pattern, then we can effectively construct the finite subtree  $T'(\mathcal{C}) \subset T(\mathcal{C})$  that is guaranteed to contain  $T_0(\mathcal{C})$ .

*Proof.* We inductively construct  $T(\mathcal{C})$ , but whenever we have a sub-path  $\mathcal{C}_i \rightarrow \dots \rightarrow \mathcal{C}_j$  of one of the three types whose length exceeded the bound  $f(\mathcal{C}_i)$ , then we stop building  $T(\mathcal{C})$  at  $\mathcal{C}_j$ , i.e.  $\mathcal{C}_j$  will not have any children. By Lemmas 3.23, 3.26, and Königs Lemma the subtree  $T'(\mathcal{C})$  will be finite and clearly contains a branch from the root to a leaf that gives us a 1-minimal pattern if one exists.  $\square$

All that remains to do is to find this bound  $f$ .

### 3.6 The Repetition Principle

We now present a combinatorial argument that can be used to give bounds on the lengths of thinning paths and quadratic paths.

**Lemma 3.28.** *Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots$  be some path in  $T(\mathcal{C})$  such the band complexes all have total complexity bounded above by  $N$ . Then there is a computable function  $r : \mathbb{N} \rightarrow \mathbb{N}$  that some unmeasured band complex occurs twice in the initial segment  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots \rightarrow \mathcal{C}_{r(N)}$ .*

*Proof.* Since the rest of the band complex stays unchanged throughout the process (e.g. If a 2-cell has trivial intersection with the bands in  $\mathcal{C}_0$  then it will always have trivial intersection with the bands), it is enough to count the combinatorial configuration of bands, items, boundaries, boundary connections, and how the 2-cells are attached to the band complex (since we are requiring them to intersect bands only in boundary connections.)

Since the total complexity is bounded by  $N$ , there are at most  $r(N)$  number of combinatorial possibilities, where  $r : \mathbb{N} \rightarrow \mathbb{N}$  is a computable function obtained by enumerating all the possibilities with total complexity  $N$ .  $\square$

**Lemma 3.29.** *Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be some path in  $T(\mathcal{C})$  Such that  $\mathcal{C}_m = \mathcal{C}_{m'}$  with  $m \neq m'$ , then this path cannot yield a 1-minimal measure.*

*Proof.* Suppose this was not the case then we have

$$p : \mathcal{C}_0 \rightarrow \dots \mathcal{C}_m \rightarrow \dots \mathcal{C}_{m'} \rightarrow \mathcal{C}_{m'+1} \dots \rightarrow \mathcal{C}_N$$

where  $\mathcal{C}_N$  has no more bands. Suppose that reversing the moves gave us a 1-minimal measure  $m$  on  $\mathcal{C}_0$ . This is impossible since reversing the moves

$$p' : \mathcal{C}_0 \rightarrow \dots \rightarrow \mathcal{C}_m \rightarrow \mathcal{C}_{m'+1} \rightarrow \dots \rightarrow \mathcal{C}_N$$

will also give us a measure  $m'$  on  $\mathcal{C}_0$  such that  $P_{m'}$  gives a free splitting, but the resulting  $P_{m'}$  will be shorter than  $P_m$ . Moreover Lemma 3.17 guarantees that  $gp(P_m) = gp(P_{m'}) = \{1\}$  which contradicts 1-minimality of  $m$ .  $\square$

These two lemmas imply

**Corollary 3.30** (The repetition principle). *There is a computable bound on the length of a path  $\mathcal{C}_i \rightarrow \mathcal{C}_{i+1} \rightarrow \dots$  in  $T_0(\mathcal{C})$  such that the total complexities of the  $\mathcal{C}_i$  are bounded above.*

### 3.6.1 Bounding the length of thinning paths

**Lemma 3.31.** *Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be sequence of band complexes obtained in a thinning process, then the total complexities of the  $\mathcal{C}_i$ 's is bounded by some computable function  $f_{\text{thin}}(\mathcal{C}_0)$ .*

*Proof.* Recall that a band complex is obtained first from a graph  $\Gamma$ , then attaching bands  $B_i$  to  $\text{int}(\Gamma^1 - \Gamma^0)$ , we call this cell complex  $Y$ . The way 2-cells are attached to  $Y$  is such that so that intersections of  $Y$  and bands are vertical subsets.

Now the elementary transformations leave invariant  $\Gamma - (\cup B_i)$ . It follows that 2-cell attaching maps only get changed within their vertical sections. Additional 2-cells may be created when we cut bases (ET1), but these new 2-cells immediately get either removed or collapsed when we collapse bands (ET4), so the total number of 2-cells is nonincreasing.

Now 2-cell attaching maps only get changed as follows: vertical sections are get mapped to points via band collapses (ET4), they get changed because of the collapse of a bigon but this doesn't change vertical lengths, and finally they can get "stretched" or collapsed in band mergers (ET7), but again this can never increase vertical lengths.

It remains to show that the number of items, bases and boundary connections is bounded. Suppose first there is a section  $\sigma$  such that  $\sigma = \mu \cup \mu'$ . Then it is clear that the thinning move leaves these numbers bounded.

Otherwise when we collapse a collapsible segment,  $J \subset \mu$  that is the union of items  $h_1, \dots, h_m$ , then we have  $\mu$ -tie  $m+1$  boundaries, which may increase the number of items by  $m$  but then we will collapse  $J$ , causing the deletion of  $m$  items as well as all the new boundary connections that were just created. So the number of items and boundary connections is nonincreasing.

To see that the number of bases is bounded, first note that the only type of thinning move that increases the number of bases is in case 3. Each time we have collapse from a separating base the number of sections increases. On the other hand the number of sections  $\sigma$  such that  $n(\sigma) > 0$  is bounded above by the  $\tau$ -complexity, so eventually we must have a section  $\sigma$  such that  $n(\sigma) = 0$ , it follows that we must have  $\sigma = \mu \cup \lambda$ . If  $\mu = \bar{\lambda}$  then these bases will never interact with the other bases and we can ignore them.

There are two now possibilities either  $\mu = \sigma = \lambda$  or w.l.o.g.  $\mu \subsetneq \sigma$ . In the first case we can perform a band merge (ET7) decreasing the number of bases and eliminating  $\sigma$ . In the other case we  $\lambda$  is half naked, in which case after at most two collapses (we never increase the number of bases) we reduce to the first case.

It follows that whenever we create a section  $\sigma$  such that  $n(\sigma) = 0$  we can remove it without increasing the number of bases. Again, the  $\tau$ -complexity gives an upper bound on the number of sections  $\sigma$  such that  $n(\sigma) \neq 0$ , and we can always remove sections such that  $n(\sigma) = 0$ , so we may assume that every base lives in a section such that  $n(\sigma) \neq 0$ , and again the  $\tau$ -complexity with the bound on the number of sections enables us to bound the number of bases.  $\square$

Note that if each section  $\sigma$  is such that  $n(\sigma) > 0$  then we have:

$$\#bases - 2\#sections \leq \tau(\mathcal{C}_0)$$

which, because the number of section such that  $n(\sigma) > 0$  is bounded by  $\tau(\mathcal{C}_0)$ , implies that

$$\#bases \leq 3\tau(\mathcal{C}_0)$$

it is enough to note that since we can always eliminate sections such that  $n(\sigma) = 0$  after at most two thinning moves and the merging of two bands and that weight zero bases are never created more than four at a time. We have that

$$\#bases \leq 3\tau(\mathcal{C}_0) + 4$$

and since none of the other quantities increase we have that  $f_{thin}(\mathcal{C}_0)$  is the total complexity of  $\mathcal{C}_0$  minus the number of bases plus  $3\tau(\mathcal{C}_0) + 4$ . The bound on the length of thinning paths is given by Corollary 3.30.

### 3.6.2 Bounding the lengths of quadratic paths

The reader is strongly encouraged to draw their own picture of this argument. In a quadratic path, in which the  $\tau$ -complexity doesn't decrease it is easy to see that entire transformations do not increase the number of bases. If we look at the attaching maps of a 2-cell  $D$ , the vertical length of  $\partial D$  can only increase in an entire transformation if  $\partial D$  contained an edge of a band containing the transfer base and the edge of the band containing the adjacent base.

When we perform the entire transformation  $\partial D$  is stretched to contain the edge of a band that contains the reincarnation of the leading base so some segment of  $\partial D$  of vertical length 2 is "stretched" to have vertical length 3. But now this segment contains subsegments segments that lie in the edges of the bands containing the two leading bases and when we perform the entire transformation again, one of these segments must be collapsed. It follows that the total complexity stays bounded above. The result now follows from Corollary 3.30

## 4 The Axial case

The axial case is by far the most complicated. The main reason for this being the failure of the repetition principle. In the works of Makanin and Razborov, this was dealt with by a study of the arising periodicity. In our context as well there will be periodicity, but we will need a sizable amount of extra machinery to exploit it.

### 4.1 Axes

For this subsection we let  $G$  be a freely decomposable group and we let  $T$  be some Bass-Serre tree representing a free splitting. Most of the discussion in this section is without proofs, but follows from elementary arguments.

First of all we have that two hyperbolic elements  $\gamma, \delta \in G$  commute if and only if their axes coincide. Let  $l \subset T$  be a biinfinite line that is the

axis of some element  $\gamma$ . Because of the free product context, we have that  $l$  is the axis of some  $\gamma'$  of minimal translation length, and we have that  $\gamma$  must be a power of  $(\gamma')^{\pm 1}$ . Let  $l_0 \subset l$  be a segment of minimal length such that

$$l = \bigcup_{n \in \mathbb{Z}} (\gamma')^n l_0$$

then  $l_0$  is called a *fundamental domain* of  $l$ .

Let  $J$  be some oriented segment of  $T$  and suppose that for some  $\gamma \in G$  we have that (up to change of orientation) a terminal subsegment of  $J$  overlaps with an initial subsegment of  $\gamma J$  then  $\gamma$  is hyperbolic and the union of translates

$$\bigcup_{n \in \mathbb{Z}} \gamma^n J$$

is exactly  $l = \text{Axis}(\gamma)$ .  $J$  contains at least one translate of  $l_0$ , the fundamental domain of  $l$ . We say that the *periodicity* of  $J$  is the number of translates of  $l_0$  that are contained in  $J$ .

**Lemma 4.1.** *Let  $G$  be freely decomposable and let  $T$  be a Bass-Serre tree for this splitting. Let  $\gamma, \delta$  be two hyperbolic elements that are not proper powers. Suppose that  $|\text{Axis}(\gamma) \cap \text{Axis}(\delta)| > \text{tr}(\gamma) + \text{tr}(\delta)$  then  $\text{Axis}(\gamma) = \text{Axis}(\delta)$ . In particular  $\gamma$  and  $\delta$  are both powers of some element  $\rho$ .*

*Proof.* Let  $J = \text{Axis}(\gamma) \cap \text{Axis}(\delta)$ . Let  $n = \text{tr}(\gamma)$  and  $m = \text{tr}(\delta)$ . First note that the action on the tree is 0-acylindrical. Take an edge  $j_0$  on the end of  $J$ . Since  $|J| \geq \text{tr}(\delta) + (\gamma) + 1$  we must have (replacing  $\delta$  or  $\gamma$  by their inverses if necessary) that  $[\delta, \gamma]j_0 = j_0$  which means by 0-acylindricity that  $[\delta, \gamma] = 1$ .

The only trees abelian groups can act minimally on are points or bi-infinite rays, it follows that the axes of  $\gamma, \delta$  coincide.

W.l.o.g.  $n \leq m$  and there are  $d, r < n \in \mathbb{N}$  such that

$$m = dn + r \tag{1}$$

Let  $j_0 \leq J$  be the initial segment of length 1 of  $J$ . If the remainder  $r = 0$  then we set  $\rho = \gamma$ .

Otherwise the remainder  $r > 0$  in (1), note that  $\gamma^{d+1}\delta^{-1}j_0 \subset J$ , it follows by the Euclidean algorithm that there is some  $\rho = \gamma^{n_1}\delta^{m_1} \dots \gamma^{n_l}\delta^{m_l}$  such that  $\rho j_0 \subset J$  and  $v = \text{tr}(\rho) = \gcd(\text{tr}(\gamma), \text{tr}(\delta))$ .

Let  $\text{tr}(\gamma) = n'v$  and  $\text{tr}(\delta) = m'v$ . It remains to show that  $\gamma$  and  $\delta$  are powers of  $\rho$ . We have:

$$\rho^{n'} \gamma^{-1} j_0 = \rho^{m'} \delta^{-1} j_0 = j_0$$

which implies (by 0-acylindricity) that  $\rho^{n'} = \gamma$  and  $\rho^{m'} = \delta$ .  $\square$

Suppose that  $H$  is another oriented segment and there is a  $\delta$  such that (as before) a terminal segment of  $H$  overlaps with an initial segment of  $\delta H$ . Suppose moreover that  $H$  is contained in  $l = \text{Axis}(\gamma)$  for some  $\gamma$ . If the intersection  $H \cap \delta H$  strictly contains at whole translate of  $l_0$  then Lemma 4.1 implies that  $\text{Axis}(\delta) = \text{Axis}(\gamma)$ .

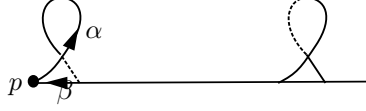


Figure 8: A tubular element associated to a band with overlapping bases

## 4.2 Tubular elements and periodic arcs

Suppose that  $G = \pi_1(\mathcal{C}, x_0)$  is freely decomposable and let  $P \subset \mathcal{C}$  be the pattern corresponding to the resolution of the action of  $G$  on  $T$ , a Bass-Serre tree representing the splitting. We now consider  $\mathcal{C}$  to be the corresponding  $\mathbb{N}$ -measured band complex.

**Definition 4.2.** Let  $\mu$  be a base in a band complex such that  $\mu \cap \bar{\mu} \neq \emptyset$ , then  $\mu, \bar{\mu}$  is called an *overlapping pair*. Moreover we denote

$$J(\mu) = \mu \cup \bar{\mu}$$

which must be a connected segment.

Let  $B$  be a band whose bases  $\mu, \bar{\mu}$  form an overlapping pair. We may assume  $I(\mathcal{C})$  to be identified with a subset of  $\mathbb{R}$  so that this identification respects lengths. We can assume that  $\mu$  is identified with  $[0, s]$  and  $\bar{\mu}$  is identified with  $[a, s + a]$  for some  $a$ .  $B$  must be orientation preserving.  $B$  is a quotient of  $\mu \times [0, 1]$  with some identifications between  $\{0\} \times \mu$  and  $\{1\} \times \mu = \bar{\mu}$ . Let  $\alpha$  be the arc corresponding to the “edge”  $\{0\} \times [0, 1]$  and let  $\beta$  be the arc corresponding to  $[0, a] \subset I(\mathcal{C})$ . The concatenation  $\alpha * \beta$  gives us a closed loop (see Figure 8). We denote by  $p$  the endpoint of  $\mu$  in  $I(\mathcal{C})$  and we denote the closed loop  $\gamma_{\mu, p} = \alpha * \beta$ .

**Definition 4.3.** This loop  $\gamma_{\mu, p} = \alpha * \beta$  as given above is called the *tubular loop associated to  $\mu$  based at  $p$* .

**Definition 4.4.** Let  $J \subset I(\mathcal{C})$  be an arc in the based band complex  $(\mathcal{C}, x_0)$ . Let  $\rho$  be a path from  $x_0$  to some  $p \in J$ . Then we say that  $J$  is  *$\rho$ -anchored at  $p$* .  $J$  has an *anchored lift*  $\tilde{J}_\rho \subset (\tilde{\mathcal{C}}, \tilde{x}_0)$  which is the lift of  $J$  containing the point  $\tilde{p}$  which is the endpoint of the lift  $\tilde{\rho}$  of  $\rho$  starting at  $\tilde{x}_0$ .

We note that if  $p \in J$  and that  $J$  is  $\rho$ -anchored at  $p$  for some  $\rho, p$ , then to any loop  $\gamma$  based at  $p$  we can assign an element  $\gamma_\rho = \rho * \gamma * \rho^{-1}$  of  $G$ . We will call this “anchored” tubular element simply a “tubular element.”

**Definition 4.5.** Let  $\mu, \bar{\mu}$  be overlapping. We denote the *translation lengths* of  $\mu$  and  $\bar{\mu}$  to be the number  $a$ . We denote this  $tr(\mu)$ .

Let  $J = J(\mu)$ , and let  $J$  be  $\rho$ -anchored at  $p$ . We wish to understand the dynamics of the action of  $(\gamma_{\mu, p})_\rho = \gamma_{\mu, p}$  on  $T$ . We denote  $\tilde{J}_\rho = \tilde{J}$ . We have the resolution  $\phi : \mathcal{C} \rightarrow T$ . Now from Figure 9 we see that  $\gamma_{\mu, p}$  acts hyperbolically on  $T$  and (in light of Section 4.1) that  $\tilde{J}$  maps onto some  $\bar{J} \subset \text{Axis}(\gamma_{\mu, p}) \subset T$  via  $\phi$ . We also immediately see that the  $tr(\mu)$  corresponds exactly to the translation length of the element  $\gamma_{\mu, p}$ . Clearly none of this depends on the choice of anchor  $\rho$ .

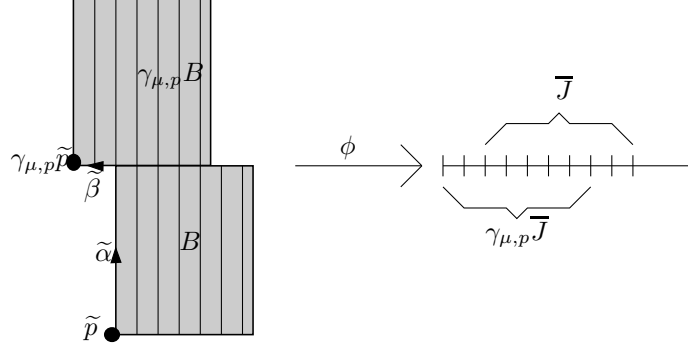


Figure 9: The action of  $\gamma_{\mu,p}$  on  $\tilde{\mathcal{C}}$  and  $T$

**Definition 4.6.** Suppose the image  $\phi(\tilde{J}_\rho) = \bar{J}$  of the anchored lift of some segment  $J \subset I(\mathcal{C}) \subset \mathcal{C}$  is mapped into some  $l = \text{Axis}(\gamma) \subset T$  via the resolution  $\phi$ . The *periodicity* of  $J$ ,  $p(J)$  is the number of translates of a fundamental domain  $l_0 \subset l$  that lie in  $\phi(\bar{J})$  (see Section 4.1). If  $p(J) > 1$  we say that  $J$  is *periodic* and we denote  $l = \text{Axis}_\rho(J)$ .

In particular we have the following:

$$\frac{|\mu|}{\text{tr}(\mu)} \geq p(\mu) > \left( \frac{|\mu|}{\text{tr}(\mu)} - 1 \right) \quad (2)$$

We now wish to extend the notion of a tubular element to bands that may not have overlapping bases. Let  $J = J(\mu)$  and let  $p \in J$ . Let  $B$  be some band with bases  $\lambda, \bar{\lambda} \subset J$  identified to  $[b, b+a], [c, c+a]$  respectively. Let  $\beta_1$  be the path in  $J$  from  $p$  to  $b$ , let  $\alpha$  be the path along the “edge” of  $B$  from  $b$  to  $c$  and let  $\beta_2$  be the path in  $J$  from  $c$  to  $p$ . We define  $\gamma_{\lambda,p} = \beta_1 * \alpha * \beta_2$  and call it the *tubular element based at  $p$  associated to  $\lambda$* . We now have a lemma that tells us to which extent we can “widen” a band.

**Lemma 4.7** (Widening Lemma). *Let  $J \subset I(\mathcal{C})$  be a periodic segment in a band complex  $(\mathcal{C}, x_0)$  with a resolving pattern  $P \subset \mathcal{C}$ . We consider  $J_\rho$  and we assume  $\rho$  ends at  $p$ . Let  $l = \text{Axis}_\rho(J)$ . Let  $\lambda, \bar{\lambda} \subset J$  and suppose the tubular element  $(\gamma_{\lambda,p})_\rho$  maps  $l$  into itself, then we can widen the band  $B_\lambda$  so that without loss of generality  $\lambda$  is coinitial with  $J$  and  $\bar{\lambda}$  is coterminal with  $J$  and the resulting band complex  $\mathcal{C}' \supset \mathcal{C}$  has isomorphic fundamental group. Moreover we can extend  $P \subset \mathcal{C}'$  to a pattern  $P \subset P' \subset \mathcal{C}'$  so that we have a  $G$ -equivariant isomorphism of trees*

$$T(P', \mathcal{C}') \rightarrow T(P, \mathcal{C})$$

*We also have that no new items are created.*

*Proof.* Let  $B = B_\lambda$  then we can attach rectangles  $B_0, B_1$  to both “sides” of  $B$  so that the resulting (wider) band  $B' = B_0 \cup B \cup B_1$  has bases  $\lambda', \bar{\lambda}'$  such that  $J = \lambda' \cup \bar{\lambda}'$ , we call the resulting band complex  $\mathcal{C}'$ , by the obvious deformation retraction we see that  $\pi_1(\mathcal{C}', x_0) \approx \pi_1(\mathcal{C}, x_0)$ . In



particular if we take the loops  $\gamma_{\lambda,p}$  and  $\gamma_{\lambda',p}$  in  $\mathcal{C}'$  we see that they are homotopic modulo  $p$ .

We also have that the pattern  $P \subset \mathcal{C} \subset \mathcal{C}'$ , but  $P$  may not be a pattern of  $\mathcal{C}'$  because  $P \cap (B_0 \cup B_1)$  consists at most of a finite collection of points. There is a natural extension of  $P$  to a pattern  $P' \subset \mathcal{C}'$ , notably the one induced by the measure, it is basically obtained drawing the missing “vertical” lines in  $B_0 \cup B_1$ . Moreover by hypothesis on the dynamics of the action of  $(\gamma_{\lambda,p})_\rho$  on  $T(P, \mathcal{C})$  we see that the tree  $T(P', \mathcal{C}')$  is  $G$ -isomorphic to  $T(P, \mathcal{C})$ .

The last statement follows by inspection.  $\square$

We finally note that a band widening does not increase the number of bases or boundary connections in a band complex (although it may change the number of boundaries.)

### 4.3 Periodic Mergers in measured band complexes

This terminology is from the orbit counting algorithm in [AHT06].

**Convention 4.8.** For this section we will consider the periodic segment  $J$  to be  $\rho$ -anchored at  $p$ , i.e.  $J = J_\rho$ . It follows that we will denote  $\tilde{J}_\rho$  simply by  $\tilde{J}$  and  $\text{Axis}_\rho(J)$  simply by  $\text{Axis}(J)$ . We shall denote tubular elements  $\rho * \gamma_{\nu,p} * \rho^{-1}$  simply as  $\gamma_\nu$ .

Suppose our band complex  $\mathcal{C}$  has two bands  $B_\mu, B_\lambda$  whose bases all lie in a periodic section  $J$ . Suppose moreover that the tubular elements  $\gamma_\mu, \gamma_\lambda$  map  $\text{Axis}(J)$  into itself. We will describe a move that replaces  $\mathcal{C}$  with a band complex  $\mathcal{C}'$  with the bands  $B_\mu$  and  $B_\lambda$  replaced by a band  $B_\nu$  such that  $J = \nu \cup \bar{\nu}$  and such that  $\text{tr}(\nu) = \gcd(\text{tr}(\lambda), \text{tr}(\mu))$ . We will also have that  $G = (\mathcal{C}, x_0) \approx (\mathcal{C}', x_0)$ , that there is a pattern  $P' \subset \mathcal{C}'$ , and that there is a  $G$ -isomorphism  $T(P, \mathcal{C}) \rightarrow T(P', \mathcal{C}')$ .

#### 4.3.1 Step 1: Adding the new band

It follows from the hypotheses and Lemma 4.7 that we can assume that the bands  $B_\lambda$  and  $B_\mu$  are “wide”, i.e.  $J = J(\lambda)$  and w.l.o.g.  $\mu$  is coinitial with  $J$  and  $\bar{\mu}$  is cofinal with  $J$ . Recall that we are studying a measured band complex corresponding to the action of a group on a Bass-Serre tree  $T$  representing a free splitting. Switching the bases for their duals if necessary, we may assume that  $\mu$  and  $\lambda$  are chosen so that they are initial segments of  $J$ . It follows that  $\gamma_\mu$  and  $\gamma_\lambda$  are powers of some element  $\gamma$ .

We now attach a band  $B_\nu$  such that  $J = \nu \cup \bar{\nu}$  and such that  $\text{tr}(\nu) = \gcd(\text{tr}(\lambda), \text{tr}(\mu))$ . With the added assumption that  $B_\nu$  is orientation preserving it is clear that there is only one way to do this. On the level of fundamental group we have added a free factor to our group. We have that there is some  $\gamma \in G$  such that  $\gamma_\mu = \gamma^m$  and  $\gamma_\lambda = \gamma^l$  and such that  $\gamma = \gamma_\mu^{m'} \gamma_\lambda^{n'}$ . We therefore want to add a 2-cell that encodes the relation  $\gamma_\nu = \gamma_\mu^{m'} \gamma_\lambda^{l'}$  so as to ensure that the fundamental group remains invariant. Unfortunately, we must be careful as we have the requirement that 2-cell attaching maps must intersect  $I(\mathcal{C})$  in a set with empty interior. The good news is that if we ensure that the anchor point  $p$  is the endpoint of

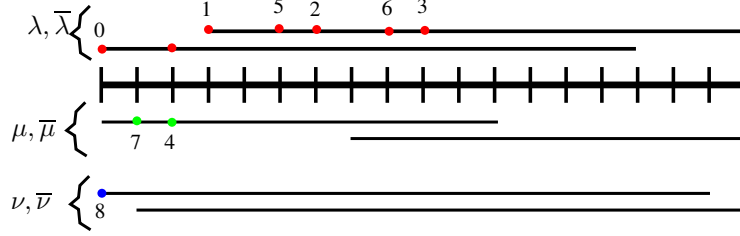


Figure 10: Bases  $\nu, \bar{\nu}, \lambda, \bar{\lambda}, \mu,$  and  $\bar{\mu}$  attached to a periodic segment.

$J$ , then we can find a curve  $\delta$  which intersects all bands only in vertical sections which is homotopic modulo  $p$  to

$$\gamma_\mu^{m'} * \gamma_\lambda^{l'} * \gamma_\nu^{-1}$$

We first need a lemma.

**Lemma 4.9.** *Let  $n, m$  be positive integers and let  $d = \gcd(n, m)$  then w.l.o.g. we have integers  $p, q \in \mathbb{Z}_{\geq 0}$  such that  $d = pn - qm$ . We have nondecreasing sequences of integers  $0 = q_0 \leq q_1 \leq \dots \leq q_n = q$  and  $0 = p_0 \leq p_1 \leq \dots \leq p_n = p$  with*

$$p_i + q_i + 1 = p_{i+1} + q_{i+1}$$

such that the following inequalities hold

$$0 \leq p_i n - q_i m \leq m + n$$

We can now state our lemma.

**Lemma 4.10.** *Let  $J, \mu, \lambda$  be as above and assume moreover that  $|J| \geq \text{tr}(\lambda) + \text{tr}(\mu)$ . Then it is possible to add a band  $B_\nu$  with  $\text{tr}(\nu) = \gcd(\text{tr}(\lambda), \text{tr}(\mu))$  and a 2-cell  $D$  so that the resulting band complex  $C'$  has the same fundamental group.  $\mathcal{C}'$  also has a natural induced pattern  $P'$  so that we have a  $G$ -isomorphism  $T(P, \mathcal{C}) \rightarrow T(P', \mathcal{C}')$ .*

*sketch.* We will only give an example of how to attach the 2-cell  $D$ . Consider Figure 10. We have bases  $\lambda$  and  $\mu$  with  $\text{tr}(\lambda) = 3$  and  $\text{tr}(\mu) = 7$ . We take a path  $\delta$  consisting of vertical segments going through first through either  $B_\lambda$  or  $B_\mu$ , and then finally through  $B_\nu$ . In Figure 10  $\delta$  is shown to start at the point labeled 0 and then intersects  $J$  again in the indicated points in the given order. Given the fact that  $\gamma_\lambda$  and  $\gamma_\mu$  commute, letting  $p$  be the point labeled 0 or 8 we have that  $\delta$  given in Figure 10 is homotopic modulo  $p$  to the closed loop

$$\gamma_\lambda^5 \gamma_\mu^{-2} \gamma_\nu^{-1}$$

If we attach a 2-cell along the loop  $\delta$  we get, On the level of the fundamental group, the Tietze transformation in which we add an element  $\gamma_\nu$  and the relation  $\gamma_\lambda^5 \gamma_\mu^{-2} \gamma_\nu^{-1} = 1$ . We call the resulting band complex  $\mathcal{C}'$ .

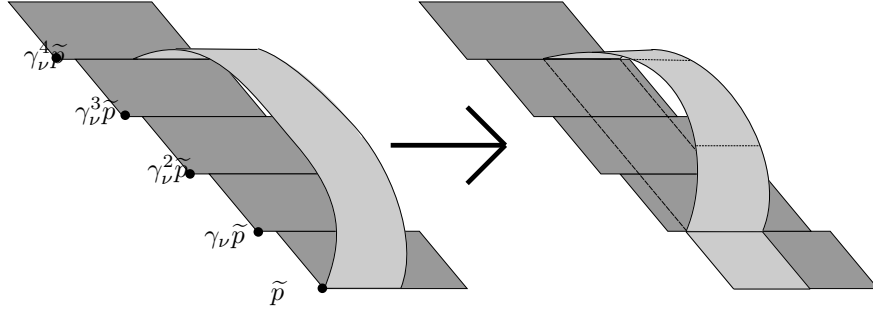


Figure 11: On the left, a piece of  $\tilde{\mathcal{C}}$ ; the lifts of  $B_\nu$  are dark and the lift of  $B_\lambda$  is light; on the right the first step of the zipping of  $B_\lambda$  into  $B_\nu$

Because  $\gamma_\lambda$  and  $\gamma_\mu$  will always commute and by Lemma 4.9, it is not hard to see how this construction can be made for arbitrary bands that satisfy the hypotheses of the Lemma.

As we saw before  $\gamma_\mu$  and  $\gamma_\lambda$  were powers of some  $\gamma$  such that  $\text{Axis}(\gamma) = \text{Axis}(J)$  and  $\text{tr}(\gamma) = \gcd(\text{tr}(\lambda), \text{tr}(\mu))$ . We now have that  $\gamma = \gamma_\nu$ . There is a natural extension  $P \subset P' \subset \mathcal{C}'$ , again the one induced by the measure, which is obtained by adding the “missing” vertical lines in  $B_\nu$ . Since we didn’t mess anything up we have that  $T(P, \mathcal{C}) \rightarrow T(P', \mathcal{C}')$  are  $G$ –isomorphic.  $\square$

### 4.3.2 Step 2: Merging Bands

For this section we want  $B_\lambda, B_\nu$  and  $J$  to be as in Section 4.3.1. Specifically let  $J$  be a periodic segment in a band complex  $\mathcal{C}'$  containing bases  $\lambda, \bar{\lambda}, \nu, \bar{\nu}$ , suppose moreover that  $J = \nu \cup \bar{\nu}$ , that  $\lambda$  is coinitial with  $J$  and that  $\gamma_\lambda = \gamma_\nu^m$ . We will describe a procedure that will eliminate the band  $B_\lambda$  while preserving the fundamental group and the splitting.

We will consider an example, the general case will follow obviously. Suppose that  $\gamma_\lambda = \gamma_\nu^4$ , then letting  $\tilde{p}$  be the anchored lift of  $p$  and by our discussion of tubular generators we must have a piece of the universal cover  $\mathcal{C}$  that looks like the example in Figure 11. We will  $G$ –equivariantly collapse the lifts of  $B_\lambda$  into the translates of  $\tilde{B}_\nu \cup \gamma_\nu \tilde{B}_\nu \cup \dots \cup \gamma_\nu^4 \tilde{B}_\nu$ , this will give a well-defined change in  $\mathcal{C}$ , the quotient. The move is just easier to visualize in the universal cover.

In our example we first subdivide  $B_\lambda$  into four bands  $B_{\lambda_1} \cup \dots \cup B_{\lambda_4}$  with  $\lambda_{i+1}$  identified with  $\bar{\lambda}_i$ . This means that any boundary connection through  $B_\lambda$  gets subdivided into four boundary connections. We now  $G$ –equivariantly identify the lifts of  $B_{\lambda_1}$  with the appropriate subsets of the lifts of  $B_\nu$  as shown on the left of Figure 11. We then repeat this for  $B_{\lambda_2}, \dots, B_{\lambda_4}$ . The resulting band complex  $\mathcal{C}''$  has one less band, although the band  $B_\nu$  has more vertical segments from 2-cell attaching maps traversing it. The pattern  $P' \subset \mathcal{C}'$  obviously induces a pattern  $P'' \subset \mathcal{C}''$  moreover it is clear that we have a  $G$ –isomorphism  $T(P', \mathcal{C}') \rightarrow T(P'', \mathcal{C}'')$ . From this discussion it is clear how one should have proceeded

for  $\gamma_\lambda = \gamma_\nu^n$  for arbitrary  $n$ .

We now summarize the discussion of this section as a Lemma.

**Lemma 4.11** (First Periodic Merging Lemma). *Let  $(C, x_0)$  be a  $\mathbb{N}$ -measured band complex such that with induced pattern  $P$  such that  $T(P, \mathcal{C})$  represents an essential free splitting of  $G = \pi_1(\mathcal{C}, x_0)$ . Let  $J$  be a periodic segment containing bases  $\lambda, \bar{\lambda}, \mu, \bar{\mu}$  with  $|J| \geq \text{tr}(\mu) + \text{tr}(\lambda)$ . Let  $\rho$  anchor  $J$  at  $p$  and suppose that  $\gamma_{\lambda, p}$  and  $\gamma_{\mu, p}$  map  $\text{Axis}_\rho(J)$  into itself. Then we can replace  $\mathcal{C}$  by a  $\mathbb{N}$ -measured band complex  $\mathcal{C}'$  with induced pattern  $P'$  such that  $\pi_1(\mathcal{C}', p) = G$  and we have a  $G$ -isomorphism  $T(P, \mathcal{C}) \rightarrow T(P', \mathcal{C}')$ .*

### 4.3.3 Another type of merger

Let  $J$  and  $L$  be maximal connected segments in  $I(\mathcal{C})$  and suppose that they are both periodic. We shall assume they are anchored, and shall not mention the paths nor the anchor points explicitly. Let  $j_0, l_0$  be fundamental domain for  $\text{Axis}(J), \text{Axis}(L)$  respectively.

**Definition 4.12.** We write  $J \ll L$  if for some  $g \in G$

$$|l_0 \cap g\text{Axis}(J)| > |j_0|$$

we write  $J \sim L$  if there is some  $g \in G$  such that

$$\text{Axis}(J) = g\text{Axis}(L)$$

**Definition 4.13.** A base  $\mu$  lying in periodic segment  $J$  is said to be *long* if  $|\mu| \geq 2|j_0| + 1$ , and *short* otherwise.

**Lemma 4.14** (Second Periodic Merging Lemma). *Suppose we have a  $\mathbb{N}$ -measured band complex  $\mathcal{C}$  with maximal (w.r.t. inclusion) segments  $J, L \subset I(\mathcal{C})$  that are periodic such that  $J \sim L$  and suppose moreover that there is a band  $B_\mu$  such that  $\mu$  is a transfer base of  $J$  and  $\bar{\mu}$  is a transfer base of  $L$ . Then by collapsing  $B_\mu$  onto its base and performing identifications in  $J$  and  $L$  we can obtain a band complex  $\mathcal{C}'$  with one less base and such that  $J$  and  $L$  are inside a common periodic segment. Moreover we have a  $G$ -isomorphism  $T(P, \mathcal{C}) \rightarrow T(P', \mathcal{C}')$  where  $P, P'$  are the patterns induced by the measures on  $\mathcal{C}$  and  $\mathcal{C}'$ .*

*Proof.* Because  $\mu, \bar{\mu}$  are long bases in  $J, L$  resp. We have that the result of the sequence of identifications described in Figure 12 has the desired properties.  $\square$

## 4.4 Periodic Mergers in Unmeasured Band complexes

So far all that we have talked about only applies to measured band complexes. In this section we will show how these moves can be applied to unmeasured band complexes. The only difference is that each move will produce a finite collection of unmeasured band complexes corresponding to all combinatorial possibilities arising from the possible measure assignments to the items.

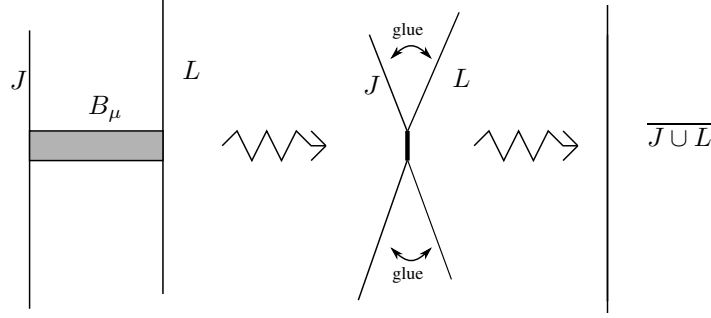


Figure 12: The merge in Lemma 4.14, the first move is to collapse the connecting band  $B_\mu$ , the second move is to isometrically identify the periodic maximal segments  $J, L$  to obtain  $\overline{J \cup L}$ .

#### 4.4.1 Some combinatorial criteria for lengths

This first lemma follows immediately from the discussion in Section 4.2

**Lemma 4.15.** *Let  $\mathcal{C}$  be an unmeasured band complex and suppose it has an overlapping pair  $\mu, \bar{\mu}$ . Then for any measure  $m$  put on  $\mathcal{C}$  such that  $T(P_m, \mathcal{C})$  gives an essential free decomposition, we must have that the segment  $\mu \cup \bar{\mu}$  is periodic.*

**Lemma 4.16.** *Let  $\bar{\mu}, \mu$  be an overlapping pair and suppose that  $\lambda, \bar{\lambda} \subset J(\mu)$ . Let  $J$  be anchored at some  $p$  and let  $\gamma_\mu$  and  $\gamma_\lambda$  be the corresponding tubular elements. If  $\gamma_\lambda$  and  $\gamma_\mu$  do not commute, then for every measure  $m$  on  $\mathcal{C}$  such that  $T(P_m, \mathcal{C})$  gives an essential free decomposition we have that*

$$|\lambda| < 2tr(\mu) + 1$$

*Proof.* Consider the action on  $T(P_m, \mathcal{C})$ . Drawing a picture we easily see that

$$\text{Axis}(\gamma_\mu) \cap \gamma_\lambda \text{Axis}(\gamma_\mu) \geq |\lambda|$$

If  $|\lambda| > 2tr(\mu) + 1$  then by Lemma 4.1 we have that the two axes  $\text{Axis}(\gamma_\mu)$  and  $\text{Axis}(\gamma_\lambda \gamma_\mu \gamma_\lambda^{-1}) = \gamma_\lambda \text{Axis}(\gamma_\mu)$  coincide. It follows that  $\gamma_\lambda$  acts on  $\text{Axis}(\gamma_\mu)$  by translation and therefore  $[\gamma_\lambda, \gamma_\mu] = 1$  – contradiction.  $\square$

This next lemma is obvious

**Lemma 4.17.** *For any measure  $m$  on  $\mathcal{C}$  and any overlapping pair  $\mu, \bar{\mu}$  we have  $|J(\mu)| = |\mu| + tr(\mu)$ .*

This next lemma gives a combinatorial criterion for the applicability of Lemma 4.7.

**Lemma 4.18.** *Let  $\mu, \bar{\mu}$  be an overlapping pair and let  $\lambda, \bar{\lambda} \subset J(\mu)$ . For any measure  $m$  on  $\mathcal{C}$ :*

1. *If  $\bar{\lambda}, \lambda$  also form an overlapping pair then  $|J(\mu)| \geq tr(\lambda) + tr(\mu)$ .*

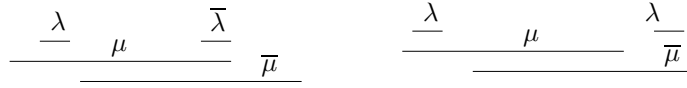


Figure 13: On the left,  $|J| \geq tr(\lambda) + tr(\mu)$ , on the right  $|J| < tr(\lambda) + tr(\mu)$ .

2. If  $\lambda, \bar{\lambda}$  do not form an overlapping pair, but they both lie in  $J(\mu)$ , then  $|J(\mu)| \geq tr(\lambda) + tr(\mu)$  if and only if either

$$(\lambda \cup \bar{\lambda}) \cap \mu \neq \emptyset \text{ or } (\lambda \cup \bar{\lambda}) \cap \bar{\mu} \neq \emptyset$$

Moreover if we have  $|J(\mu)| < tr(\lambda) + tr(\mu)$  then we must also have  $|\lambda| < tr(\mu)$ .

*Proof.* Let  $J(\mu) = \mu \cup \bar{\mu}$  and we define  $J(\lambda)$  analogously if  $\lambda, \bar{\lambda}$  form an overlapping pair. We immediately see that  $|J(\mu)| = |\mu| + tr(\mu)$  and that  $|\mu| \geq tr(\mu)$ .

(1.) Suppose that  $\lambda, \bar{\lambda}$  form an overlapping pair as well, then we have that

$$\begin{aligned} |J(\mu)| &= |J(\mu)| - |J(\lambda)| - |J(\mu) \cap J(\lambda)| \\ &= |\mu| + tr(\mu) + |\lambda| + tr(\lambda) - |J(\mu) \cap J(\lambda)| \\ &\geq |J(\mu) \cap J(\lambda)| \end{aligned}$$

so in all cases the inequality holds.

(2.) In this case we have  $|J(\mu)| = |\mu| + tr(\mu)$  and the inequality holds only if  $tr(\lambda) \leq |\mu|$ . This is equivalent to the given criterion. The second fact is obvious.  $\square$

The two criteria given in the Lemma are purely combinatorial. See Figure 13 for an illustration of item 2.

What we want is a version of the First periodic merger for unmeasured band complexes similar to the one given in Lemma 4.11. We will want to use it when we have an overlapping pair  $\mu, \bar{\mu}$  and bases  $\lambda, \bar{\lambda} \subset \mu \cup \bar{\mu}$  such that the tubular elements  $\gamma_\mu, \gamma_\lambda$  commute. The trick given in the next section will enable us to determine whether there exists some  $u \in G$  such that  $\gamma_\mu = u^n, \gamma_\lambda = u^m$  and if so will find it.

#### 4.4.2 A Bulitko Trick

If  $G$  has decidable word problem, then it is also decidable whether some finite subset  $S \subset G$  generates an abelian subgroup of  $G$ . What is not quite so clear is what the abelian subgroup  $\langle S \rangle$  will be, specifically we would like to be able to decide if  $\langle S \rangle$  is cyclic and find a generator. Although this is still not known to be decidable, we have will obtain a slightly weaker decidability result which will be perfectly applicable to tubular elements.

In [Bul70] Bulitko showed that given a solution  $f$  to system of equations over a free group and a cyclically reduced word  $p$  it was possible to produce another solution  $f'$  such that there is some  $d \in \mathbb{N}$  such that for any occurrence of  $p^n, n \in \mathbb{Z}$  in the solutions,  $|n| \leq d$ . To prove our

theorem, we need a Bulitko Lemma for free products. Let  $A, B$  be groups and denote by  $\widetilde{A \cup B}$  the set of *reduced words* in  $A \cup B$  i.e. elements of  $\widetilde{A \cup B}$  are exactly normal forms of elements of  $A * B$ . We use  $*$  to denote concatenation.

**Definition 4.19.** Let  $w = a_1 b_2 \dots a_{n-1} b_n$  and  $p = \alpha_1 \dots \beta_m$  be in  $\widetilde{A \cup B}$ . Suppose moreover that  $p$  is not a proper power, not elliptic and cyclically reduced. We denote the  $p$ -stable normal form of  $w$  to be the product

$$u_0 p^{n_0} u_1 p^{n_1} \dots p^{n_l} u_{l+1}$$

which is *graphically equal* to  $w$  that satisfies:

- $u_0$  lies in  $(\widetilde{A \cup B} * p^{\pm 1}) - (\widetilde{A \cup B} * p^{\pm 1} * p^{\pm 1} * \widetilde{A \cup B})$ .
- $u_i$  lies in  $(p^{\pm 1} * \widetilde{A \cup B} * p^{\pm 1}) - (\widetilde{A \cup B} * p^{\pm 1} * p^{\pm 1} * \widetilde{A \cup B})$ , for  $1 \leq i \leq l$ .
- $u_k$  lies in  $(p^{\pm 1} * \widetilde{A \cup B}) - (\widetilde{A \cup B} * p^{\pm 1} * p^{\pm 1} * \widetilde{A \cup B})$ .

The terms  $n_1, \dots, n_l$  are called the  $p$ -stable exponents of periodicity. For convenience we will denote the  $p$ -stable normal form as a tuple  $(u_0, n_1, \dots, n_l, u_{l+1})$ .

**Lemma 4.20.** For a given  $w$  and  $p$ , the  $p$ -stable normal form is unique.

We also note that the existence of such an  $f$  means that  $G$  acts on the Bass-Serre tree  $T$  of  $A * B$ , if  $f : G \rightarrow A * B$  is essential, then at least one of tracks we get from the resolution of the action is essential. The Bulitko lemma is usually a statement about solutions to equations, however it translates directly into an analogous statement about homomorphisms.

**Lemma 4.21** (Bulitko Variant for free products). Suppose  $f : \langle X \mid R \rangle \rightarrow A * B$  is homomorphism from a finitely presented group given as a mapping

$$x_i \mapsto w_i$$

where  $x_i \in X$  and  $w_i \in \widetilde{A \cup B}$ . Let the  $w_i$  have  $p$ -stable normal forms

$$(u_0, n_0, \dots, n_l, u_{l+1})$$

then there is a homomorphism  $f' : \langle X \mid R \rangle \rightarrow A * B$  such that  $f'(x_i) = w'_i$  and the  $w'_i$  have  $p$ -stable normal forms

$$(u_0, n_0^+, \dots, n_l^+, u_{l+1})$$

where the  $|n_j^+|$  are bounded above by some computable function of  $X$  and  $R$ .

The argument for the case of free groups is easily adapted to free products so we omit the proof. Now, being sneaky, we can get the following result:

**Theorem 4.22.** Let  $G$  be a f.p. group with decidable word problem and let  $\gamma, \delta$  be two elements of  $G$  such that  $[\gamma, \delta] = 1$ . Then there is an algorithm that terminates and either:

- (a) outputs an element  $u \in G$  and integers  $n, m$  such that  $u^n = \gamma$  and  $u^m = \delta$ ; or

(b) the algorithm proves that if  $\gamma$  and  $\delta$  lie in a cyclic subgroup, then there is no homomorphism  $f : G \rightarrow A * B$  such that the images  $f(\gamma)$  and  $f(\delta)$  are hyperbolic.

*Proof.* Suppose there is a map  $f : G \rightarrow A * B$  such that  $\gamma, \delta$  are sent to hyperbolic elements. Then we can change the presentation  $\langle X \mid R \rangle$  of  $G$  to  $\langle X \cup \{\gamma, \delta\} \mid R' \rangle$  so that  $\gamma, \delta$  now lie in a generating set. Let  $X' = X \cup \{\gamma, \delta\}$ , and apply Lemma 4.21 to  $\langle X' \mid R' \rangle$ .

Suppose that  $\gamma = u^n$  and  $\delta = u^m$  for some  $u \in G, n, m \in \mathbb{Z}$ . Suppose that  $\gamma$  or  $\delta$  is sent to a hyperbolic element, then so must  $u$ . We therefore have that for  $p$  (replacing it by a proper root, if necessary) which is the cyclically reduced “core” of  $f(u)$  we have that  $\gamma$  and  $\delta$  have non trivial  $p$ -stable exponents of periodicity, moreover these exponents of periodicity give upper bounds for  $|n| - 2$  and  $|m| - 2$ .

On the other hand by Lemma 4.21 we can replace  $f$  by some  $f'$  with bounded  $p$ -stable exponents of periodicity, but even better: if  $\gamma$  and  $\delta$  were sent to hyperbolic elements via  $f$ , the same is true via  $f'$ , so  $u$  must also be sent to a hyperbolic element, so the equalities  $\gamma = u^n$  and  $\delta = u^m$  reveal themselves via translation lengths. Upper bounds for  $|n| - 2$  and  $|m| - 2$  as given in the statement of the Theorem are therefore given by Lemma 4.21.

So we have a bound on  $n, m$  such that  $u^n = \gamma$  and  $u^m = \delta$ , it follows that  $\gamma^{n'} = \delta^{m'}$  where e.g.  $n' = \text{lcm}(n, m)/n$ . Now using our solution to the word problem for  $G$  we can check for all the finite possibilities for  $n', m'$  whether  $\gamma^{n'} = \delta^{m'}$ . In which case we have

$$n' = \frac{m}{\gcd(m, n)} \text{ and } m' = \frac{n}{\gcd(m, n)}$$

and by the Euclidean algorithm there are  $r, s \in \mathbb{Z}$  such that  $1 = rn' + sm'$  it follows that

$$u = \gamma^r \delta^s$$

and, again by our solution to the word problem, we can check whether  $u^n = \gamma$  and  $u^m = \delta$ .

If all this searching yielded no results, then (b) of the statement of the Theorem must hold.  $\square$

#### 4.4.3 The first Periodic Merger for unmeasured band complexes

If we want to perform a periodic merger, we must first add a band, and then add a 2-cell, this corresponds to the sequence of presentations:

$$\langle X \mid R \rangle \rightsquigarrow \langle X \cup \{x'\} \mid R \rangle \rightsquigarrow \langle X \cup \{x'\} \mid R \cup \{x = \gamma_\lambda^{n'} \gamma_\mu^{m'}\} \rangle$$

which we want to be a Tietze transformation.

**Lemma 4.23** (First Periodic Merger for unmeasured band complexes). *Suppose that we have an overlapping pair  $\mu, \bar{\mu}$  and a pair of bases  $\lambda, \bar{\lambda}$  that satisfy either premises 1. or 2. of Lemma 4.18. If the tubular elements  $\gamma_\mu$  and  $\gamma_\lambda$  commute then Theorem 4.22 either tells us that there is no measure  $m$  that can be put on  $\mathcal{C}$  so that  $T(P_m, \mathcal{C})$  gives a free spitting*



or gives us integers  $r, s$  and an element  $u \in G$  such that  $u = \gamma_\lambda^{r'} \gamma_\mu^{s'}$  and  $\gamma_\lambda = u^r, \gamma_\mu = u^s$ .

In the latter case from  $\mathcal{C}$  we can produce finitely many band complexes  $\mathcal{C}_1, \dots, \mathcal{C}_q$  such that they all have the same fundamental group as  $\mathcal{C}$ , but all have one less band than  $\mathcal{C}$ . Specifically we will have added a band  $B_\delta$  such that  $\gamma_\delta = u$  and removed the bands  $B_\mu$  and  $B_\lambda$ . Moreover if  $\mathcal{C}$  admits a measure  $m$  such that  $T(P_m, \mathcal{C})$  gives a free splitting, then there is some  $i \in \{1, \dots, q\}$  such that  $\mathcal{C}_i$  with an induced measure  $m_i$  such that  $T(P_{m_i}, \mathcal{C}_i)$  also gives a free splitting.

Finally we have that any 1-minimal measure  $m'$  on  $\mathcal{C}_i$  induces a 1-minimal measure  $\hat{m}$  on  $\mathcal{C}$  by reversing the moves.

*Proof.* Because  $\lambda, \bar{\lambda}$  satisfy the criteria of Lemma 4.18 we have that  $J(\mu) \geq tr(\lambda) + tr(\mu)$  for any measure  $m$  on  $\mathcal{C}$ .

Now because  $\mu, \bar{\mu}$  form an overlapping pair for any measure  $m$  on  $\mathcal{C}$  such that  $T = T(P_m, \mathcal{C})$  gives a free decomposition we must have that  $\gamma_\mu$  and  $\gamma_\lambda$  act hyperbolically on  $T$ , moreover in this case they must lie in a common cyclic subgroup. We now apply Theorem 4.22 to the elements  $\gamma_\mu$  and  $\gamma_\lambda$ .

Obviously if it is determined that if  $\gamma_\mu$  and  $\gamma_\lambda$  lie in a cyclic subgroup, then there is no homomorphism  $f : G \rightarrow A * B$  such that the images  $f(\gamma_\mu)$  and  $f(\gamma_\lambda)$  are hyperbolic. Then there is clearly no measure  $m$  on  $\mathcal{C}$  such that  $T(P_m, \mathcal{C})$  gives a free splitting.

Otherwise we have found an element  $u$  and integers  $r', s'$  such that  $\gamma_\lambda^{r'} \gamma_\mu^{s'} = u$ . We can widen the band  $B_\lambda$  so that w.l.o.g.  $\lambda, \bar{\lambda}$  is coinital, coterminal with  $J(\mu, )$  where. It is easy to see that there are finitely many possible resulting combinatorial band complexes.

We now add the new band  $B_\delta$  such that  $\bar{\delta} \cup \delta = J(\mu, \lambda)$  and we add the 2-cell whose boundary is sent into vertical segments, as in Lemma 4.10, giving relation  $\gamma_\delta = \gamma_\lambda^{r'} \gamma_\mu^{s'}$ . As pointed out earlier the new band complex has the same fundamental group. There are again finitely many combinatorial band complexes that can be obtained from adding a band and a 2-cell of vertical length  $r' + s'$  to  $\mathcal{C}$ .

Finally we zip the the bands  $B_\lambda$  and  $B_\mu$  onto  $B_\delta$  as described in Section 4.3.2. Again it is easy to see that there are finitely many possible resulting combinatorial band complexes as we have explicit bounds the vertical length of the 2-cell attaching maps and the number of bands.

So far we have created finitely many combinatorial band complexes  $\mathcal{C}_1, \dots, \mathcal{C}_q$ . If  $\mathcal{C}$  has a measure  $m$  such that  $T(P_m, \mathcal{C})$  gave a free splitting, then we could apply Lemma 4.11 (i.e. The First Periodic Merging Lemma) to get a measure  $m'$  and  $\mathcal{C}'$ . The unmeasured band complex corresponding to  $\mathcal{C}'$  must be some  $\mathcal{C}_i$  for  $i \in \{1, \dots, q\}$ .

Note finally that Convention 2.25 ensures that any measure  $m$  put on one of the resulting  $\mathcal{C}'_i$ s induces a measure  $\hat{m}$  on  $\mathcal{C}$ .  $\square$

Again note that the premises of this next lemma are combinatorial:

**Lemma 4.24.** *Let  $\mu, \bar{\mu}$  and  $\lambda, \bar{\lambda}$  be overlapping pairs that are coinital. Suppose w.l.o.g. that  $|J(\mu)| > |J(\lambda)|$  and suppose moreover that  $|\lambda| > tr(\mu)$  then if  $\mathcal{C}$  has a measure  $m$  such that  $T(P_m, \mathcal{C})$  gives a free splitting then we must have that the tubular elements  $\gamma_\mu, \gamma_\lambda$  commute.*

*Proof.* By hypothesis  $J(\lambda) \cap J(\mu) = J(\lambda)$ . Now for any measure on  $\mathcal{C}$  we have  $|J(\lambda)| = |\lambda| + \text{tr}(\lambda)$ . If  $|\lambda| > \text{tr}(\mu)$  then the result immediately follow by Lemma 4.1.  $\square$

We note that an equivalent condition is that  $\lambda \cap (\mu \cap \overline{\mu})$  has nonempty interior. We can now define a new elementary transformation. Since this transformation removes a band each time it is performed it can only be applied finitely many times along a path.

**Definition 4.25 (ET8: Periodic merger).** If  $\mathcal{C}$  is superquadratic and the carrier  $\mu$  overlaps with its dual, then for any other base  $\lambda$  such that  $J(\mu) \supset \lambda, \overline{\lambda}$  we do one of the following:

1. If  $\lambda, \overline{\lambda}$  forms an overlapping pair and  $J(\lambda)$  is coinital with  $J(\mu)$ . Then if  $\lambda$  is also a leading base and  $\lambda$  has nonempty interior with in  $\overline{\mu} \cap \mu$  then if  $\gamma_\lambda$  and  $\gamma_\mu$  don't commute we declare  $\mathcal{C}$  to be *unsuitable* (indeed, Lemma 4.24 precludes this possibility.) Otherwise we perform the periodic merger given in Lemma 4.23.
2. If the tubular elements  $\gamma_\lambda, \gamma_\mu$  commute, and  $\lambda, \overline{\lambda}$  satisfy one of the conditions given in Lemma 4.18 then we perform the periodic merger given in Lemma 4.23.

This next lemma follows from Lemma 4.16 and says what happens when a periodic merger *cannot* be applied.

**Lemma 4.26.** *Suppose that  $\mu$  is a carrier base that overlaps with its dual and bases  $\lambda, \overline{\lambda}$  satisfy the premises of Lemma 4.18, but that the tubular elements  $\gamma_\mu, \gamma_\lambda$  do not commute, then for any measure  $m$  on  $\mathcal{C}$  we must have that  $|\lambda| \leq 2\text{tr}(\mu)$ .*

## 4.5 Restricted Elimination Processes

Perhaps before we say anything further we should give some motivation for this section: ultimately our proof of the algorithmic constructibility of  $T_0(\mathcal{C})$  follow from the algorithmic constructibility of a  $J$ -restricted elimination trees  $T_0(\mathcal{C}, J)$ .

Let  $J \subset I(\mathcal{C})$  be a union of closed sections. Recall that we had an embedding  $i : I(\mathcal{C}) \hookrightarrow \mathbb{R}$ . We can redefine  $i$  on so that all the (interiors of) bases contained in  $J$  are mapped to the left of all the bases contained  $I(\mathcal{C}) - J$ .

In this case if we start our elimination process each leading base will lie in  $J$ . Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be some sequence of band complexes obtained by applying thinning moves and entire transformations, then we have a natural inclusions  $I(\mathcal{C}_i) \supset I(\mathcal{C}_{i+1})$  induced by considering how bands correspond to partial isometries of  $\mathbb{R}$ . Let  $J$  be some segment in  $\mathcal{C}$  then we denote  $J_i$  to be  $J \cap I(\mathcal{C}_i)$ .

**Definition 4.27.** The arborecence of band complexes  $T(J, \mathcal{C})$  is a tree that is constructed by starting with  $\mathcal{C}$  and assuming that all the bases in  $J$  are to the left of the bases in  $I(\mathcal{C}) - J$ . We ensure moreover that we only apply thinning moves if some base in  $J$  contains a free segment, otherwise we apply entire transformations. For every  $\mathcal{C}_i \in T(J, \mathcal{C}) - \partial T(J, \mathcal{C})$ ,  $J \cap \mathcal{C}_i$  is nonempty. The leaves of the infinite tree  $T(J, \mathcal{C})$  are band complexes

$\mathcal{C}_j$  such that  $J \cap \mathcal{C}_j = \emptyset$ , i.e. all the bases in  $J$  have been moved onto  $I(\mathcal{C}) - J$ .

Let  $m$  be some measure on  $\mathcal{C}$  such that  $T(P_m, \mathcal{C})$  gives a free splitting. The argument used in Lemma 3.17 gives an induced path in  $T(J, \mathcal{C})$  from the root  $\mathcal{C}$  to a leaf. We denote  $T_0(J, \mathcal{C})$  the minimal subtree of  $T(J, \mathcal{C})$  containing all the paths corresponding to 1-minimal measures.

**Definition 4.28.** A  $J$ -restricted elimination process is the construction of a finite subtree  $T'(J, \mathcal{C}) \subset T(J, \mathcal{C})$  which is guaranteed to contain  $T_0(J, \mathcal{C})$ .

#### 4.5.1 The excess argument

We will develop some machinery meant to deal with the so called superquadratic case. The arguments presented are more or less standard in the theory of generalized equations and are essentially due to Razborov. *Throughout this section  $J$  denotes some union of closed section of  $I(\mathcal{C})$ .* Our goal is to bound the length of paths  $p : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots \mathcal{C}_j$  in  $T(J, \mathcal{C})$  that aren't unsuitable.

**Definition 4.29.** A union of closed sections  $J \subset I(\mathcal{C})$  is called *unthinnable* if for each item  $h \in J$  we have  $\gamma(h) \geq 2$  and is called *superquadratic* if it is unthinnable and has an item  $h$  such that  $\gamma(h) > 2$ .

**Definition 4.30.** If  $J$  is superquadratic then we can write  $J = Q(J) \cup SQ(J)$  where  $Q(J)$  is the union of items  $h$  such that  $\gamma(h) = 2$  and  $SQ(J)$  is the union of the items such that  $\gamma(h) > 2$ .  $SQ(J)$  is called the *strictly superquadratic part*.

Clearly for any measure on  $\mathcal{C}$ ,  $|J| = |Q(J)| + |SQ(J)|$ .

**Definition 4.31.** Let  $\Delta = \{\delta_1, \dots, \delta_k\}$  be set of bases that lie in  $J$  but whose duals lie  $I(\mathcal{C}) - J$ . We call these *connectors*.

Obviously if a connector  $\delta \in \Delta$  is ever a carrier in a  $J$ -restricted elimination process then the complexity  $\tau(J)$  will decrease in the subsequent band complexes. Let  $p : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots \rightarrow \mathcal{C}_N$  be some path in  $T(J, \mathcal{C})$  and let  $C(p)$  denote the set of bases that are carriers in  $p$  and  $T(p)$  be the set of transfer bases in  $p$  i.e. the set of bases that are carried. Suppose moreover that the complexities  $\tau(J) = \tau(J_i)$  for all  $0 \leq i \leq N$ . Then in particular no base in  $\Delta$  is ever a carrier.

**Definition 4.32.** Let  $p, T(p)$  and  $C(p)$  be as above. We denote

$$I(p) = \bigcup_{\lambda \in T(p), C(p)} \lambda$$

We denote  $I_j(p) = I(p) \cap \mathcal{C}_j$ .

We also immediately see that we have containments  $J_j \supset I_j(p) \supset I_{j+1}(p)$ . Now let  $\Delta' = \Delta \cap T(p)$ . And let  $Z \subset I(p)$  denote the union of segments in  $I(p)$  covered by bases in  $\Delta$ . Let  $Z_1, \dots, Z_k$  denote the connected segments of  $Z$  covered by bases in  $\Delta$  that are never carried in  $p$ . Let

$$I' = I(p) - Z$$

and let  $Q(I')$  be the quadratic part of  $I'$  and let  $SQ(I)$  denote the strictly superquadratic part of  $I$ .

**Lemma 4.33.** *There is a recursive function  $f_{rep}$  with values in  $\mathbb{Z}_{\geq 0}$  such that for any 1-minimal measure on  $\mathcal{C}$ ,  $I(p)$  and  $J$  given as above we can get a bound*

$$|Q(I')| \leq f_{rep}(\mathcal{C}, J, I(p)) \left( |SQ(I(p))| + \sum_{\delta \in \Delta'} |\delta| + \sum_{i=1}^k |Z_i| \right)$$

*Proof.* Subdividing bases, we arrange so that  $I'$  is a union of closed sections. Since  $I'$  is quadratic by the repetition principle we can construct the finite elimination tree  $T_0(I', \mathcal{C})$ . The leaves not corresponding to unsuitable paths correspond to paths in  $T(I', \mathcal{C})$  where all the bases in  $I'$  are moved onto  $SQ(I(p))$  or onto bases in  $Z$ . For each branch from a root to a leaf going backwards enables us to bound above the lengths of the bases in  $I'$  in terms of  $|SQ(J)|$ ,  $|\delta|$  and  $|Z_i|$  where  $\delta \in \Delta$ ,  $i = 1, \dots, k$ . Since there are finitely many such leaves we can take  $f_{rep}$  to be the maximum.  $\square$

**Corollary 4.34.** *For any 1-minimal measure on  $\mathcal{C}$  we have*

$$|I(p)| \leq (f_{rep} + 1) \left( |SQ(I(p))| + \sum_{\delta \in \Delta'} |\delta| + \sum_{i=1}^k |Z_i| \right)$$

*Proof.* Its enough to observe that  $|I(p)| \leq |SQ(I(p))| + |Z| + |I'|$ .  $\square$

**Definition 4.35.** Let  $\mathcal{C}$  be a band complex with  $I \subset I(\mathcal{C})$  some superquadratic union of items. Let  $m$  be some measure on  $\mathcal{C}$ . Then we define the *excess*

$$\psi_m(\mathcal{C}, I) = \sum_{h \in H(I)} (\gamma(h) - 2)|h| \quad (3)$$

where  $H(J)$  is the set of items contained in  $J$ . We will usually omit mention of the measure in the subscript. For  $I(p)$  as above, we define the *J-relative excess* to be

$$\psi_m^*(\mathcal{C}, J, I(p)) = \psi_m(\mathcal{C}, I(p)) + \sum_{\delta \in \Delta'} |\delta| + \sum_{i=1}^k |Z_i|$$

**Lemma 4.36.** *For every measure on  $\mathcal{C}$  we have the upper bound*

$$|I(p)| \leq (f_{rep}(\mathcal{C}, J, I(p)) + 1) \psi_m^*(\mathcal{C}, J, I(p))$$

*Proof.* This follows immediately since  $\psi_m(\mathcal{C}, I(p)) \geq |SQ(I(p))|$ .  $\square$

**Lemma 4.37.** *Let  $m$  be a measure on  $\mathcal{C}$ . And let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  where  $J$  is superquadratic. Suppose moreover that for each  $\mathcal{C}_i$  we have  $\tau(J_i) = \tau(J)$ . Let  $I(p)$  be as defined above. We have the following:*

1. *If  $I(p)$  is superquadratic then  $\psi(\mathcal{C}, I(p)) > 0$*
2.  *$\tau(J_i) = \tau(J)$  we have*

$$\psi(\mathcal{C}_i, I_i(p)) = \psi(\mathcal{C}_{i+1}, I_{i+1}(p))$$

3. In each  $\mathcal{C}_i$  in  $p$  we have that the quantity

$$\sum_{\delta \in \Delta'} |\delta| + \sum_{i=1}^k |Z_i|$$

is constant. It follows that

$$\psi^*(\mathcal{C}_i, J_i, I_i(p)) = \psi^*(\mathcal{C}_{i+1}, J_{i+1}, I_{i+1}(p))$$

4. For each  $i$ , let  $\lambda_i$  be the longest base in  $\mathcal{C}_i$  contained in  $T(p) \cup C(p)$ , let  $L_i = |\lambda_i|$ , and let  $N$  be the number of bases in  $\mathcal{C}$ . Then

$$\psi^*(\mathcal{C}_i, J_i, I_i(p)) \leq 3N^2 L_i$$

*Proof.* 1. This is obvious since  $\psi(\mathcal{C}_i, I_i(p)) > |SQ(I_i(p))|$ .

2. We first note that entire transformations will always send the bases in  $I_i(p)$  into  $I_{i+1}(p)$ . Keeping track of the summands of (3) show that the quantity remains constant.

3. Since bases in  $\Delta$  are never carriers we must have that the quantity  $\sum_{\delta \in \Delta'} |\delta|$  remain fixed. Moreover the carrier in  $\mathcal{C}_i$  cannot lie in  $Z_i$  because  $Z_i$  is covered by bases of  $\Delta$  that are *never carried* in  $p$ . It follows that  $|Z_i|$  also remain invariant.

4. It is enough to notice that  $I_i(p)$  is always exactly covered by the bases in  $T(p) \cup C(p)$ . It follows that for each  $\delta$  in  $\Delta'$  and  $Z_i, i = 1, \dots, k$  we have  $|\delta|, |Z_i| \leq N L_i$ . We also have that  $|\Delta'|, k \leq N$ . Finally we have that  $\psi(\mathcal{C}_i, I_i(p)) < N |I_i(p)| \leq N^2 L_i$ . It follows that  $\psi^*(\mathcal{C}_i, J_i, I_i(p)) < 3N^2 L_i$ .  $\square$

**Definition 4.38.** Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  satisfying the premises of Lemma 4.37. Then for any measure  $m$  on  $\mathcal{C}_0$  we define the quantity

$$\psi^*(p) = \psi^*(\mathcal{C}, J, I(p))$$

when  $p$  is understood we will simply denote it  $\psi^*$ .

Immediately we get:

**Corollary 4.39.** Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  satisfying the premises of Lemma 4.37 where  $\mathcal{C}$  has  $N$  bases. Then for any 1-minimal measure  $m$  on  $\mathcal{C}$  we have

$$|I(p)| \leq (f_{rep}(\mathcal{C}, J, I(p)) + 1) \psi^*$$

let  $\lambda_i$  be the longest base  $T(p) \cup C(p)$  in  $\mathcal{C}_i$  and let  $L_i = |\lambda_i|$  then we have

$$|I(p)| \leq (f_{rep}(\mathcal{C}, J, I(p)) + 1) 3N^2 L_i$$

**Definition 4.40.** Let  $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a sequence of band complexes we denote the sequence

$$\mathcal{C}[i, j] : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1} \rightarrow \dots \mathcal{C}_j$$

We say that  $\mathcal{C}[i, j]$  and  $\mathcal{C}[k, l]$  are *disjoint* if  $[i, j] \cap [k, l]$  are disjoint as intervals in  $\mathbb{R}$ .

**Definition 4.41.** Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  satisfying the premises of Lemma 4.37. Let  $T(p)$  and  $C(p)$  denote the set of transfer bases and carriers respectively. A  $T - C$  cycle is a sequence  $\mathcal{C}[i, j]$  such that each base in  $T(p)$  is carried at least once. Moreover for each  $\lambda$  in  $C(p)$  there are  $i < k < l < j$  such that the following occur, either:

1.  $\lambda$  is the carrier in  $\mathcal{C}_k$ , but it doesn't overlap with its dual. Some base  $\delta$  is carried by  $\lambda$  and  $\delta$  is a leading base again in  $\mathcal{C}_l$ ; or
2.  $\lambda$  is the carrier in  $\mathcal{C}_k$  and  $\lambda$  overlaps with its dual  $\bar{\lambda}$ . For some  $k \leq s < l$ ,  $\lambda$  is always the carrier base throughout  $\mathcal{C}[k, s]$  but is not the carrier in  $\mathcal{C}_{s+1}$ . Then  $\lambda$  is the carrier base again in  $\mathcal{C}_l$

**Lemma 4.42.** Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  satisfying the premises of Lemma 4.37. Let  $T(p)$  and  $C(p)$  denote the set of transfer bases and carriers respectively. Let  $\mathcal{C}[i, j]$  be a  $C - T$  cycle. Let  $\lambda \in T(p)$  and let  $k, l$  be as in Definition 4.41. Let  $D_k$  be the length  $|\lambda|$  in  $\mathcal{C}_k$ . Then for any measure  $m$  on  $\mathcal{C}_0$ , and induced measures on the subsequent band complexes we have

$$|I_i(p)| - |I_j(p)| \geq |I_k(p)| - |I_l(p)| \geq D_k/2$$

*Proof.* W.l.o.g. we oriented  $I(\mathcal{C})$  so that entire transformations move everything to the right. Suppose first that the carrier  $\lambda$  in  $\mathcal{C}_k$  doesn't intersect its dual. Then we have  $(\lambda) > |\lambda| = D_k$ . If  $\delta$  is carried by  $\lambda$  then for  $\delta$  to be a leading base again in  $\mathcal{C}_l$  we must have cut off a set of length at least  $D_k$  from  $I_k(p)$  to get  $I_l(p)$ . The inequality therefore holds.

Suppose now that the carrier  $\lambda$  in  $\mathcal{C}_k$  overlaps with its dual. Let  $s$  be as given in Definition 4.41. Let  $L_\lambda = |\lambda|$  in  $\mathcal{C}_k$ . There are two possibilities:

- $\lambda$  is the carrier throughout  $\mathcal{C}[k, s]$  and after a certain number of repetitions at least  $D_k/2$  was cut out from  $I_k(p)$ .
- Before we can cut  $D_k/2$  from  $I_k(p)$  another base  $\rho$  becomes the carrier.

We consider the second possibility. Let  $D_{s+1}$  denote  $|\lambda|$  in  $\mathcal{C}_{s+1}$ , we have  $D_{s+1} > D_k/2$ . Note also that although  $|\lambda|$  decreased in passing from  $\mathcal{C}_k$  to  $\mathcal{C}_{s+1}$ ,  $tr(\lambda)$  remained constant, also  $\lambda$  and  $\bar{\lambda}$  still form an overlapping pair which means that  $tr(\lambda) < D_{s+1}$ .

Suppose first that the carrier  $\rho$  in  $\mathcal{C}_{s+1}$  did not overlap with its dual. First of all we know that  $|\rho| \geq D_{s+1} > D_k/2$ , and when  $\lambda$  is a leading base again we will know that we have cut at least  $|\rho| > D_k$  from  $I_k(p)$  in passing to  $I_l(p)$  from the first part of the proof applied to  $\rho$ .

Suppose now that  $\rho$  overlaps with its dual. We have two possibilities. First if  $J(\rho) \subset J(\lambda)$ , since  $\rho$  is a carrier we have that  $|\rho| \geq |\lambda| > tr(\lambda)$  and since  $|J(\rho)| = |\rho| + tr(\rho)$  and  $J(\rho) \subset J(\lambda)$  we have  $|J(\rho) \cap J(\lambda)| = |J(\rho)| > tr(\rho) + tr(\lambda)$ , which means that we can perform a periodic merger thus decreasing the  $\tau$ -complexity –contradiction.

The second case is that  $J(\lambda) \subset J(\rho)$ . If  $D_{s+1} = |\lambda| > tr(\rho)$  (this can be seen combinatorially) again we can do a periodic merger. It follows that  $tr(\rho) \geq |\lambda| > D_k/2$  which means that once  $\lambda$  is a leading base again, we would have cut at least  $D_k/2$  from  $I_k(p)$  in passing to  $I_l(p)$ .  $\square$

**Definition 4.43.** Let  $p : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  be a path in  $T(J, \mathcal{C})$  satisfying the premises of Lemma 4.37 where  $\mathcal{C}$  has  $N$  bases. We set

$$B(p) = 6N^2(f_{rep}(\mathcal{C}, J, I(p)) + 1)$$

**Lemma 4.44.** Let  $\mathcal{C}$  be a band complex and let  $m$  be any 1-minimal measure on  $\mathcal{C}$ . Let

$$p : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$$

be a path satisfying the premises of Lemma 4.36. Let  $C(p)$  and  $T(p)$  denote the set of carriers and transfer bases in  $p$  respectively. Then it is impossible for  $p$  to contain  $B(p) + 1$  disjoint  $C - T$  cycles  $\mathcal{C}[i, j]$ . Specifically such a path is unsuitable.

*Proof.* Let  $L_i$  denote the length of the longest base  $\lambda_1 \in T(p) \cup C(p)$  in  $C_i$ . By Corollary 4.39

$$|I(p)| < B(p) \frac{L_k}{2} \quad (4)$$

for all  $k = 0, 1, \dots$ . Note that

$$L_0 \geq L_1 \geq L_2 \geq \dots \quad (5)$$

also note that the only time when the length of a base decreases is when it is a carrier. Note furthermore that if  $\mathcal{C}[i, j]$  is a  $C - T$  cycle then since each base in  $T(p)$  gets carried the base  $\lambda$  such that  $|\lambda| = L_i$  in  $\mathcal{C}_i$  must lie in  $C(p)$ .

By definition of a  $C - T$  cycle  $\lambda$  will eventually be a carrier in  $\mathcal{C}[i, j]$ . By Lemma 4.42, (4), and (5) we have that

$$|I_i(p)| - |I_j(p)| > \frac{L_i}{2} > \frac{|I(p)|}{B(p)}$$

which means that each time a cycle occurs at least  $\frac{|I(p)|}{B(p)}$  gets cut out from  $|I(p)|$ . For a 1-minimal measure on  $\mathcal{C}$  this cannot happen more than  $B(p) + 1$  times. It follows that  $p$  is unsuitable.  $\square$

As a corollary we have this important fact:

**Lemma 4.45.** Suppose we take a subtree  $T'(\mathcal{C}, J)$  of  $T(\mathcal{C}, J)$  such that

- For every subpath  $q : \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  no  $C - T$  cycle occurs more than  $B(q) + 1$  times.
- There is some computable function  $p'$  such that for each  $\mathcal{C}_i$  in  $T'(\mathcal{C}, J)$  such that the its carrier  $\mu$  overlaps with its dual  $\mu$  is not the carrier base more than  $p'(\mathcal{C}_i, J)$  times in a row.

Then  $T'(\mathcal{C}, J)$  is finite.

*Proof.* Since  $T(\mathcal{C}, J)$  has finite branching so must  $T'(\mathcal{C}, J)$ . Suppose towards a contradiction that  $T'(\mathcal{C}, J)$  was infinite, by König's Lemma we must have that  $T'(\mathcal{C}, J)$  has an infinite branch. Let  $q : \mathcal{C}_r \rightarrow \mathcal{C}_{r+1} \rightarrow \dots$  be this infinite branch. W.l.o.g. we can take a tail of  $q$  such that each base that carries is a carrier infinitely often and each base that is carried is carried infinitely often.

The first possibility is that  $C(q) = \{\lambda\}$ , but by hypothesis the computable function  $p'(\mathcal{C}_r, \lambda)$  gives us a computable bound on the number of times in a row  $\lambda$  can be the carrier and therefore implies that  $q$  is finite – contradiction.

The second possibility is that  $C(q)$  contains at least two bases. But since each base that carries is a carrier infinitely often and each base that is carried is carried infinitely often we must have infinitely many  $C - T$  cycles – contradiction.  $\square$

**Proposition 4.46.** *Suppose we have a computable function  $p(\mathcal{C}, \mu)$  which bounds the number of times in a row the carrier  $\mu$  of  $\mathcal{C}$  can be the carrier in some path  $p : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots$  that is not unsuitable. Then we can algorithmically construct the  $J$ -restricted elimination tree  $T_0(\mathcal{C}, J)$ .*

*Proof.* In light of the discussion in Sections 3.5 and 3.6 we see that to construct  $T_0(\mathcal{C}, J)$  all that remains to be shown is that we can algorithmically bound the length of paths  $q : \mathcal{C}_r \rightarrow \mathcal{C}_{r+1} \rightarrow \dots$  in which all the  $J_i \subset \mathcal{C}_i$  are superquadratic, the complexities  $\tau(J_i)$  remain constant, i.e. paths that satisfy the premises of Lemma 4.37 and such the  $q$  is not unsuitable.

We will construct  $T'_0(\mathcal{C}, J)$  as the subtree of  $T(\mathcal{C}, J)$  which contain no paths  $q : \mathcal{C}_r \rightarrow \mathcal{C}_{r+1} \rightarrow \dots$  with more than  $B(q) + 1$   $C - T$  cycles or such that the carrier  $\lambda_r$  of  $\mathcal{C}_r$  is a carrier more than  $p(\mathcal{C}_r, \lambda)$  times in a row. By Lemma 4.44 and by assumption on  $p(\mathcal{C}, J)$ , we have that  $T'_0(\mathcal{C}, J)$  contains  $T_0(\mathcal{C}, J)$ . By Lemma 4.45  $T'_0(\mathcal{C}, J)$  is finite, so we can simply set  $T'_0(\mathcal{C}, J) = T_0(\mathcal{C}, J)$ .  $\square$

Of course letting  $J = I(\mathcal{C})$  implies our result. This should be sufficient to motivate the next section.

## 4.6 Bounding Periodicity

All that therefore remains to be shown is that there is a computable function  $p(\mathcal{C}, \lambda)$  as given in Proposition 4.46.

Let  $m$  be a 1-minimal measure on a band complex  $\mathcal{C}$  and suppose that the carrier  $\mu$  overlaps with its dual. Then  $J(\mu)$  is a periodic segment, in particular in  $T(P_m, \mathcal{C})$  the axis of  $\gamma_\mu = tr(\mu)$  is covered by segments  $j_\mu$  such that  $|j_\mu| = tr(\gamma_\mu)$  and we have

$$|J(\mu)| \leq p(\mathcal{C}, m, \mu) tr(\mu)$$

Suppose that after we do an entire transformation  $\mu$  is a carrier again, then it's easy to see that  $tr(\mu)$  is unchanged and that  $J(\mu)$  was shortened by exactly  $tr(\mu)$ . It follows that this  $p(\mathcal{C}, m, \mu)$  gives an upper bound on the number of times in a row  $\mu$  can be a carrier. We will show how to recursively find upper bound  $p(\mathcal{C}, \mu)$  of  $p(\mathcal{C}, m, \mu)$ , for any 1-minimal  $m$  in terms of  $\mathcal{C}$ .

**Lemma 4.47.** *Suppose  $J \subset \mathcal{C}$  is a superquadratic union of closed sections. Then we can build a maximal finite subtree  $T_{overlap}(\mathcal{C}, J) \subset T_0\mathcal{C}, J)$  such that for any path*

$$p : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_1 \rightarrow \dots \rightarrow \mathcal{C}_l$$



from the root to a leaf, either the carrier  $\lambda_l$  of  $\mathcal{C}_l$  overlaps with its dual and  $\bar{\lambda}_l$  is not contained in another base,  $p$  is unsuitable, or the complexity  $\tau$  decreased.

*Proof.* In particular the carriers bases in the band complexes in the interior of  $T_{\text{overlap}}(\mathcal{C}, J)$  are never carriers twice in a row. It follows by Lemma 4.46, that this tree is finite.  $\square$

#### 4.6.1 Bringing a band complex into periodic block form: the tree $T_p(\mathcal{C})$

We will define another type of elimination tree. Suppose first that the carrier  $\mu$  overlaps with its dual, suppose moreover that there is no base  $\lambda$  such that  $\lambda \supset \mu$  or  $\bar{\mu}$ . We construct the tree  $T_p(\mathcal{C})$  as follows (see also Figure 14):

1. We apply ET8 as often as possible, either  $\mathcal{C}$  is unsuitable or it isn't.
2. We attach a hanging band  $B_\delta$  (ET6) such that  $\delta = \mu \cup \bar{\mu}$ , we call the new band complex  $\mathcal{C}'$ . We arrange so that the embedding  $I(\mathcal{C}') \rightarrow \mathbb{R}$  extends  $I(\mathcal{C}) \rightarrow \mathbb{R}$  so that  $\bar{\delta}$  is at the far right.
3.  $\delta$  is now a carrier base so we do an entire transformation, i.e. we transfer  $\mu, \bar{\mu}$  and all bases contained in  $J(\mu)$  onto  $\bar{\delta}$  and we collapse the initial lonely subsegment of  $\delta$ . The resulting band complex is still superquadratic.
4. We now declare the segment to the right covered by  $J(\mu)$  to be a *periodic block*.
5. We proceed with the  $(I(\mathcal{C}) - J(\mu))$ -restricted elimination process. By Lemma 4.47 we can construct the finite tree  $T_{\text{overlap}}(\mathcal{C}, I(\mathcal{C}) - J(\mu))$  such that at each leaf  $l$  either the carrier base  $\mu_1$  overlaps with its dual and there is no base properly containing  $\bar{\mu}_1$  (or the complexity decreases, or the leaf is at the end of an unsuitable path). We create a new periodic block  $J(\mu_1)$  as in 3. We continue by growing the tree  $T_{\text{overlap}}(\mathcal{C}, I(\mathcal{C}) - (J(\mu) \cup J(\mu_1)))$  at the leaf  $l$  and eventually get more periodic block at the leaves.
6. Continuing in this fashion we finally get a tree  $T_P(\mathcal{C})$  such that all the band complexes at the leaves are either unions of periodic blocks or the ends of unsuitable paths.

**Definition 4.48.** Let  $\tau_I(\mathcal{C}) = \tau(J(\mu_1) \cup \dots \cup J(\mu_k))$  be the sum of the complexities of the periodic blocks. Let  $\tau_P(\mathcal{C}) = \tau(\mathcal{C}) - \tau_I(\mathcal{C})$ .

**Lemma 4.49.** *If  $\mathcal{C}$  is superquadratic, creating a periodic block doesn't change the complexity  $\tau(\mathcal{C})$ . Moreover the  $\tau$ -complexity of each section that is a periodic block is at least 1. This means that  $\tau_p$  decreases when we create a new periodic block.*

*Proof.* Attaching the hanging band  $B(\delta)$  creates a new section  $\sigma = \bar{\delta}$  such that  $n(\sigma) = 0$  (see Definition 3.4) and adds a base  $\delta$  to  $\mathcal{C}$  which increases the complexity by at most 1, since adding a base  $\delta = J(\mu)$  doesn't decrease the number of sections. Now since  $\mu$  was a carrier and there are no bases

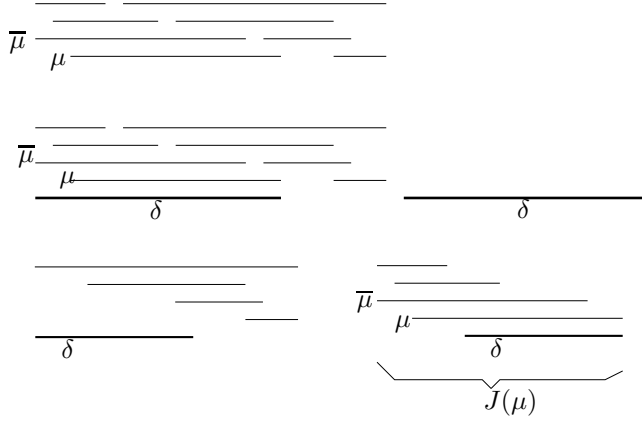


Figure 14: Creating an inert periodic section

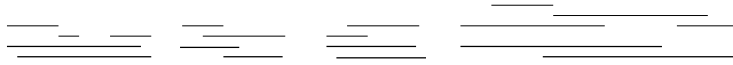


Figure 15: A band complex in periodic block form

containing  $\mu$  or  $\bar{\mu}$  we have that  $\delta$  is a carrier. We can first move  $\mu$  onto  $\sigma$ ,  $n(\sigma)$  remains 0 whereas the complexity of the section that contained  $J(\mu)$  decreases by 1. Any other transfers via  $\delta$  increase  $n(\sigma)$  by 1 and decrease the complexity of the section that contained  $J(\mu)$  by 1, so after we have transferred  $\mu, \bar{\mu}$  and all bases contained in  $J(\mu)$  the total change in complexity is 0. Since  $\sigma$  contains at least three bases we have  $\tau(\sigma) > 0$  so the inequality follows.  $\square$

**Corollary 4.50.**  *$T_P(\mathcal{C})$  is finite and all its leaves are either unsuitable or consist uniquely of periodic blocks, in either case the  $\tau$ -complexity never increased. Moreover any 1-minimal measure on  $\mathcal{C}$  induces a path to some leaf in  $T_P(\mathcal{C})$ .*

*Proof.* Whenever we create a new periodic blocks or whenever a base is transferred to a periodic blocks, the complexity  $\tau_p$  decreases. We also know how to bound thinning and quadratic paths. In all cases along any path from  $\mathcal{C}$  the complexity  $\tau_P$  eventually decreases to zero and the process stops or (by Corollary 4.47) we arrive at an unsuitable  $\mathcal{C}_j$ . The last part follows as in the proof of Lemma 3.17.  $\square$

#### 4.6.2 Periodic Hierarchies

**Definition 4.51.** A band complex  $\mathcal{C}$  that consists only of periodic blocks is said to be in *periodic block form* (see Figure 15.)

It is clear from the construction that each periodic block is some  $J(\mu)$ , for  $\mu$  some base overlapping with its dual. From Lemma 4.26 we have:

**Lemma 4.52.** *Let  $\mathcal{C}$  be in periodic block form and suppose it is no longer possible to perform the periodic merger ET8 in any of the blocks. Then for each base  $\lambda$  inside a periodic block  $J(\mu)$  and for any 1-minimal measure we have that:*

- *If both  $\lambda$  and  $\bar{\lambda}$  lie inside  $J(\mu)$  then  $|\lambda| \leq 2tr(\mu)$ .*
- *If  $|\lambda| > 2tr(\mu)$  then  $\bar{\lambda} \subset J(\mu')$  for some  $\mu' \neq \mu, \bar{\mu}$ .*

**Definition 4.53.** A band whose bases lie in different periodic blocks is called a *periodic connector*. We say a base  $\lambda \subset J(\mu)$  is *long* w.r.t.  $\mu$  if  $|\lambda| > 2tr(\mu)$ .

**Lemma 4.54.** *Suppose that  $J$  is in periodic block form and has a measure  $m$ . The relation  $\ll$  on the set of periodic blocks defined as follows:*

- *if there is a periodic connector  $B_\lambda$  between periodic blocks  $J(\mu_1)$  and  $J(\mu_2)$  with  $\lambda$  long w.r.t.  $\mu_1$  then we set  $J(\mu_1) \ll J(\mu_2)$ .*
- *If  $J(\mu_1) \ll J(\mu_2)$  and  $J(\mu_2) \ll J(\mu_1)$  we write  $J(\mu_1) \sim J(\mu_2)$ .*

*is a partial order.*

This next lemma gives us a criterion if  $J(\mu_1) \sim J(\mu_2)$ .

**Lemma 4.55.** *Let  $J(\mu_1), J(\mu_2)$  be periodic blocks with  $B_\lambda$  the periodic connector giving the equivalence  $J(\mu_1) \sim J(\mu_2)$ . Let  $\lambda \subset J(\mu_1)$  and let  $\bar{\lambda} \subset J(\mu_2)$ . Let  $p \in \lambda$ , let  $\alpha$  be a vertical path in  $B_\lambda$  going from  $p$  to  $p'$ . Let  $J(\mu_1)$  be  $\rho$ -anchored at  $p$  and let  $J(\mu_2)$  be  $\rho * \alpha$ -anchored at  $p'$  (see Definition 4.4). Then the anchored tubular elements  $(\gamma_{\mu_1, p})_\rho$  and  $(\gamma_{\mu_2, p'})_{\rho * \alpha}$  must commute.*

*sketch.* This follows immediately by the action on  $T(P_m, \mathcal{C})$ . In particular  $(\gamma_{\mu_1, p})_\rho$  and  $(\gamma_{\mu_2, p'})_{\rho * \alpha}$  have the same axis by Lemma 4.1 and the fact that the connector is  $B_\delta$  has long bases w.r.t. both periodic blocks.  $\square$

**Definition 4.56.** Let  $\mathcal{C}$  be in periodic block form. Any partial order  $\ll'$  on the set of periodic blocks is called a *periodic hierarchy*. If  $J(\mu_1) \sim J(\mu_2)$  via some hierarchy  $\ll'$  but the corresponding tubular elements; as given in Lemma 4.55; do not commute, then we say  $\ll'$  is *inadmissible*. Otherwise  $\ll'$  is *admissible*.

It is clear that there are only finitely many hierarchies definable on  $J$  in periodic block form: it's enough to decide if the bases of a connector are long or not in their respective blocks. This motivates the following terminology.

**Definition 4.57.** Let  $J$  be in periodic block form with an admissible hierarchy  $\ll'$  and let  $B_\delta$  be a connector. A base  $\delta$  is said to be  $\ll'$ -*long* in  $J(\mu_1)$  if  $B_\delta$  induces some  $J(\mu_1) \ll' J(\mu_2)$ .

If  $\mathcal{C}$  admits a 1-minimal measure, then the induced hierarchy must be one of the finite choices of admissible periodic hierarchies. We now have a second periodic merger lemma:

**Lemma 4.58** (Second Periodic Merger for unmeasured band complexes). *Suppose we have  $\mathcal{C}$  in periodic block form with an admissible periodic hierarchy  $\ll'$ . Suppose  $J(\mu) \sim J(\lambda)$  via some periodic connector  $B_\delta$ . Then we can produce finitely many unmeasured band complexes  $\mathcal{C}_1, \dots, \mathcal{C}_m$  by*

collapsing  $B_\delta$  and considering all the possible combinatorial possibilities that can arise in making the identifications in  $J(\mu)$  and  $J(\lambda)$ . For all these new band complexes we have  $\pi_1(\mathcal{C}_i, x_0) = \pi_1(\mathcal{C}, x_0)$ . Moreover if  $\mathcal{C}$  admits a measure  $m$  such that  $T(P_m, \mathcal{C})$  gives a free splitting then for some  $\mathcal{C}_i$  with induced measure  $m_i$  we have that  $T(P_{m_i}, \mathcal{C})$  also gives a free splitting.

Finally we have that any 1-minimal measure  $m'$  on  $\mathcal{C}_i$  induces a 1-minimal measure  $\hat{m}$  on  $\mathcal{C}$  by reversing the moves.

*Proof.* We wish to perform the identifications given in Lemma 4.14 (i.e. the Second Periodic Merging Lemma). Obviously there are only finitely many possible combinatorial possibilities that can arise. The rest of the statement as well as the invariance of the fundamental group follow immediately from Lemma 4.14.  $\square$

Obviously one can immediately follow this move by a periodic merger of the first type so that the resulting band complexes have two less bands and each periodic block is still some  $J(\mu)$ . This motivates the following lemma.

**Lemma 4.59.** *Performing a second periodic merger, followed by a first periodic merger on two inert periodic sections decreases the  $\tau$ -complexity by 2.*

*Proof.* By hypothesis, both sections must contain at least three bases and hence have  $\tau$ -complexity at least 1. The second periodic merger causes the removal of two base and one section, so this leaves the  $\tau$ -complexity unchanged. Moreover the resulting section has at least four bases in it. The first periodic merger forces the removal of two bases and therefore decreases the  $\tau$ -complexity by 2.  $\square$

### 4.6.3 Bounding Periodicity of a maximal periodic block

Let  $J \subset I(\mathcal{C})$  be in periodic block form with periodic blocks  $J(\mu_1), \dots, J(\mu_m)$ . And let  $\ll'$  be some fixed periodic hierarchy. We will now describe an effective process to find upper bounds for the periodicity  $p(\mathcal{C}, m, \mu_i)$  of a maximal periodic block  $J(\mu_i)$  (as given in the beginning of Section 4.6) for any 1-minimal measure on  $\mathcal{C}$ .

Let  $J(\mu)$  be maximal w.r.t.  $\ll'$ . Using the second periodic merger for unmeasured band complexes, as well as the first, we may assume that the  $\ll'$ -maximal block  $J(\mu)$  is unique in its  $\sim$ -class, indeed suppose that  $J(\mu)$  was merged with some other  $J(\mu')$  then it is clear after applying the first periodic merger that  $p(\mathcal{C}, m, \mu) \leq p(\mathcal{C}, m, \mu'')$  where  $\mu''$  is the new overlapping pair covering both  $J(\mu)$  and  $J(\mu')$  so replacing  $\mu$  with  $\mu''$  still gives an upper bound.

If ET8 can no longer be applied, by the maximality assumption it follows that every base lying in  $J(\mu)$  will be short w.r.t.  $J(\mu)$  for any 1-minimal measure that gives  $\ll'$ .

**Definition 4.60.** Let  $J(\mu)$  be a maximal periodic block we say:

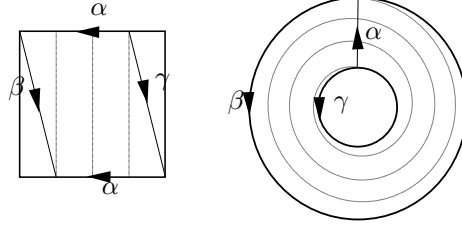


Figure 16: Part of a naked segment on a band. The annulus is obtained by cutting the band along the lines  $\beta$  and  $\gamma$ . The dotted lines represent part of a track, the translation length of the tubular element is 1.

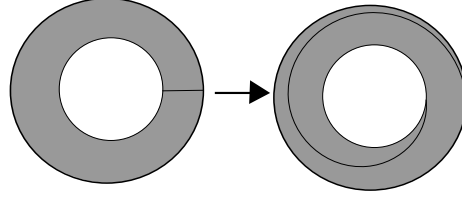


Figure 17: The image of an arc after performing a Dehn twist in an annulus.

- if  $I \subset J(\mu)$  is connected and such that each  $x \in I$  lies in at least three bases and  $J$  is maximal in that respect, then  $I$  is a *strongly covered segment*.
- if there is a boundary connection  $c$  that  $\mu$ -ties points  $x$  and  $x'$  in  $J(\mu)$ , then if the interval  $[x, x']$  doesn't lie in a strongly covered segment then we say that  $[x, x']$  is *weakly covered*.
- a connected  $N \subset J(\mu)$  which is maximal in that its intersection with strongly and weakly covered segments has empty interior is called a *naked segment*.

**Lemma 4.61.** *Let  $m$  be a 1-minimal measure on  $\mathcal{C}$  in periodic block form, with  $J(\mu)$  a maximal block that is unique in its class. Then for any naked segment  $N \subset J(\mu)$  we have*

$$|N| \leq tr(\mu)$$

*Proof.* Suppose this was not the case. Let  $P$  be the pattern associated to  $m$ . If  $J(\mu)$  has a naked segment  $N$  then this means that the cell complex  $\mathcal{C}$  has an embedded annulus with a very particular pattern, see Figure 16. If  $|N| > tr(\mu)$  we can apply a Dehn twist (see Figure 17). Since this Dehn twist is actually a homeomorphism of the cell complex, which preserves the band complex, all that we have done is produce a new pattern  $P'$  with  $Gp(P) = Gp(P')$ . Moreover this new pattern is still essential. The resulting pattern is such that we can get a new measure  $m'$  such that  $|N|_{m'} = |N|_m - tr(\mu)$ . This contradicts 1-minimality.  $\square$

**Corollary 4.62.** *Let  $J(\mu)$  be as above and let  $m$  be a 1-minimal measure. Then*

$$p(\mathcal{C}, m, \mu) \leq 2B + C + N' = p(\mathcal{C}, \mu)$$

where  $B$  is the number of bases lying in  $J(\mu)$ ,  $C$  is the number of boundary connections through  $\mu$  and  $N'$  is the number of naked segments, moreover these three numbers in no way depend on the choice of measure.

*Proof.* First it is clear that weakly covered segments have length exactly  $tr(\mu)$ , moreover all the other bases in  $J(\mu)$  are short, and therefore have length at most  $2tr(\mu)$ , and by Lemma 4.61 naked segments have length at most  $(\mu)$ . Since the length of  $J(\mu)$  is the sum of the lengths of its strongly covered segments, its weakly covered segments and its naked segments the result follows.  $\square$

#### 4.6.4 Computing $p(\mathcal{C}_1, \lambda_1)$ : auxiliary trees

Suppose that at some band complex  $\mathcal{C}_1$  in our elimination tree  $T_0(\mathcal{C})$  we have that the carrier and its dual  $\lambda_1, \bar{\lambda}_1$  form an overlapping pair. We wish to compute  $p(\mathcal{C}_1, \lambda_1)$  in order to do this we will need to construct an *auxiliary tree*  $T_{aux}(\mathcal{C}_1, \lambda_1)$ . This tree of band complexes will have *conventional* edges corresponding to steps in the normal (restricted) elimination process as well as *auxiliary* edges which will appear in auxiliary subtrees.

This finite auxiliary tree will have a collection of leaves that are in periodic block form with periodic hierarchies such that the periodic block corresponding to  $J(\lambda_1)$  is maximal. Using Corollary 4.62 and taking the maximum over all these leaves will give us the bound  $p(\mathcal{C}_1, \lambda_1)$ .

**Convention 4.63.** Suppose that at some point the periodic blocks  $J(\mu_i)$  and  $J(\mu_j)$  are merged, then in subsequent incarnations we will denote the resulting periodic block as either  $J(\mu_i)$  or  $J(\mu_j)$ .

We now construct the tree  $T_{aux}(\mathcal{C}_1, \lambda_1)$ . The construction will be inductive. Assume first that we are able to construct auxiliary subtrees as in step 4. below. Let  $\mathcal{C}_1$  and  $\lambda_1$  be as above:

1. For each leaf of the tree  $T_P(\mathcal{C}_1)$  that is in periodic block form and for each admissible periodic hierarchy  $\mathcal{C}'_1$  we draw an *auxiliary* edge from  $\mathcal{C}_1$  to  $\mathcal{C}'_1$ .
2. If the periodic block  $J(\lambda_1)$  is maximal in  $\mathcal{C}'_1$  then  $\mathcal{C}'_1$  is a leaf.
3. If the periodic block  $J(\lambda_1)$  is not maximal, then we consider the restricted elimination process which transfers the bases of  $\mathcal{C}'_1 - J(\lambda_1)$  onto  $J(\lambda_1)$ .
4. Suppose that in this restricted elimination process there is band complex  $\mathcal{C}_2$  such that the carrier base  $\lambda_2$  overlaps with its dual, then we construct the auxiliary tree  $T_{aux}(\mathcal{C}_1, \lambda_1, \lambda_2)$ . This tree will be finite and some of its leaves will be in periodic block form with the periodic block  $J(\lambda_2)$  maximal, using Corollary 4.62 and taking the maximum over all such leaves gives us a value  $p_{aux}(\mathcal{C}_2, \lambda_2)$  which we use to bound the number of times in a row  $\lambda_2$  can be carrier.

5. By Lemma 4.45 the tree  $T_{aux}(\mathcal{C}_1, \lambda_1)$  is finite. Moreover all its leaves that aren't unsuitable are in periodic block form. We set  $p(\mathcal{C}_1, \lambda_1)$  to be the maximum taken over all leaves where  $J(\lambda_1)$  is a maximal periodic block.

In this construction we see that there are auxiliary subtrees. We define the *level* of a band complex  $\mathcal{C}'$  in  $T_{aux}(\mathcal{C}_1, \lambda_1)$  to be the number of auxiliary edges in a path from  $\mathcal{C}'$  to  $\mathcal{C}_1$ . We now explain how to construct the auxiliary subtree  $T_{aux}(\mathcal{C}_k, \lambda_1, \dots, \lambda_k)$ :

1. Suppose we are performing a restricted elimination process in the tree  $T_{aux}(\mathcal{C}_{k-1}, \lambda_1, \dots, \lambda_{k-1})$  and that at the band complex  $\mathcal{C}_k$  is at level  $k$ . In particular this is a restricted elimination process where we want to move the bases of  $\mathcal{C}'_{k-1} - (J(\lambda_1) \cup \dots \cup J(\lambda_{k-1}))$  onto  $J(\lambda_1) \cup \dots \cup J(\lambda_{k-1})$ , where  $\mathcal{C}'_{k-1}$  is connected to  $\mathcal{C}_{k-1}$  by an auxiliary edge. Suppose that the carrier  $\lambda_k$  of  $\mathcal{C}_k$  overlaps with its dual.
2. We start constructing  $T_{aux}(\mathcal{C}_k, \lambda_1, \dots, \lambda_k)$  by setting  $\mathcal{C}_k$  as a root, then for each leaf of  $T_P(\mathcal{C}_k)$  which is in periodic block form and each admissible periodic hierarchy  $\mathcal{C}'_k$  we draw an auxiliary edge from  $\mathcal{C}_k$  to  $\mathcal{C}'_k$ .
3. If any of the periodic blocks  $J(\lambda_1), \dots, J(\lambda_k)$  are maximal in  $\mathcal{C}'_k$  then  $\mathcal{C}'_k$  is a leaf.
4. Otherwise at  $\mathcal{C}'_k$  we start the restricted elimination process which moves the bases of  $\mathcal{C}'_k - (J(\lambda_1) \cup \dots \cup J(\lambda_k))$  onto  $J(\lambda_1) \cup \dots \cup J(\lambda_k)$ .
5. Suppose that in this elimination process we have a band complex  $\mathcal{C}_{k+1}$  such that its carrier  $\lambda_{k+1}$  overlaps with its dual. Then we construct the auxiliary tree  $T_{aux}(\mathcal{C}_{k+1}, \lambda_1, \dots, \lambda_{k+1})$ , this tree will have leaves in periodic block form where  $J(\lambda_{k+1})$  a maximal periodic block, using Corollary 4.62 and taking the maximum over such leaves gives us a value  $p'(\mathcal{C}_{k+1}, \lambda_{k+1})$ .
6. Assuming the auxiliary subtrees can be constructed, Lemma 4.46 ensures that the restricted elimination process at level  $k$  terminates. The leaves that aren't unsuitable consist of band complexes where all the bases have been moved onto  $J(\lambda_1) \cup \dots \cup J(\lambda_k)$ . These are in periodic block form, moreover one of these periodic blocks must be maximal. In all cases we see that we can compute  $p'(\mathcal{C}_k, \lambda_k)$ .

**Lemma 4.64.** *The level of a vertex in  $T_{aux}(\mathcal{C}_1, \lambda_1)$  can never exceed  $\tau(\mathcal{C})$ .*

*Proof.* Note on one hand that as we follow any path in  $T_{aux}(\mathcal{C}_1, \lambda_1)$ , the  $\tau$ -complexity never increases. On the other hand consider a vertex  $\mathcal{C}'_j$  in  $T_{aux}(\mathcal{C}_j, \lambda_1, \dots, \lambda_j)$  at level  $j$ . We are trying to move all the bases onto the union of periodic blocks  $J(\lambda_1) \cup \dots \cup J(\lambda_j)$ . Assume first there were no periodic mergers, then there are at most  $j$  periodic blocks and since each periodic section contributes at least 1 to the  $\tau$ -complexity we cannot have more than  $j$  sections.

Suppose now that there were some periodic mergers between the periodic blocks so that  $J(\lambda_1) \cup \dots \cup J(\lambda_j)$  in fact consisted of only  $j - k$  blocks. This means that there were  $k$  periodic mergers among the  $J(\lambda_1) \cup$

$\dots \cup J(\lambda_j)$  and by Lemma 4.59 each periodic merger decreases the  $\tau$ -complexity by 2. This means that  $j - k$  cannot exceed  $\tau(\mathcal{C}) - 2k$  or that  $j \leq \tau(\mathcal{C}) - k$ .  $\square$

**Proposition 4.65.**  $T_{aux}(\mathcal{C}_1, \lambda_1)$  is finite.

*Proof.* The proof is by induction on the depth of the auxiliary subtrees. Suppose first that we create an auxiliary tree  $T_{aux}(\mathcal{C}_k, \lambda_1, \dots, \lambda_k)$  where  $\mathcal{C}_k$  is at level  $\tau(\mathcal{C}) - 1$  and the carrier  $\lambda_k$  overlaps with its dual. Then by the analysis of Lemma 4.64, the union of the periodic segments  $J(\lambda_1) \cup \dots \cup J(\lambda_{k-1})$  “count” for at least  $\tau(\mathcal{C}) - 1$ . In other words  $\tau(\mathcal{C}_k - J(\lambda_1) \cup \dots \cup J(\lambda_{k-1})) = 1$ , so at the end of each auxiliary edge originating at  $\mathcal{C}_k$  we have that the band complex consists exactly of the periodic sections  $J(\lambda_1), \dots, J(\lambda_k)$ . It follows that one of them must be maximal so the tree  $T_{aux}(\mathcal{C}_k, \lambda_1, \dots, \lambda_k)$  is finite by step 3. of the construction.

Now suppose that all auxiliary trees at level  $j > n$  could be constructed, then by step 6. of the construction we have that trees at level  $n$  can also be constructed.  $\square$

So we have constructed some tree and have some value for  $p(\mathcal{C}_1, \lambda_1)$ . All that remains to show is that this bound will work as advertised.

#### 4.6.5 The bound $p(\mathcal{C}_i, \lambda_1)$ works: Induced paths

Let  $m$  be a 1-minimal measure on a band complex  $\mathcal{C}$  as we saw the measure  $m$  on  $\mathcal{C}$  induces a sequence of band complexes

$$\mathcal{C} = \mathcal{C}_0(m) \rightarrow \mathcal{C}_1(m) \rightarrow \dots$$

obtained by applying the appropriate thinning moves or entire transformations.  $\mathcal{C}_i(m)$  has an 1-minimal measure  $m(\mathcal{C}_i(m))$ . This sequence corresponds to a path in the elimination tree  $T_0(\mathcal{C})$

**Definition 4.66.** We say band complex  $\mathcal{C}_i(m)$  is  $\lambda$ -special if its carrier  $\lambda$  overlaps with its dual.

Given  $\mathcal{C}_i(m)$  that is special and denoting  $m'_i = m(\mathcal{C}_i(m))$  and  $\mathcal{C}_i = \mathcal{C}_i(m)$ , the measure  $m'_i$  induces a path

$$\mathcal{C}_i \rightarrow \mathcal{B}_1(m'_i) \rightarrow \mathcal{B}_2(m'_i) \rightarrow \mathcal{B}_N(m'_i)$$

where  $\mathcal{B}_N$  is in periodic block form and the induced measure  $m(\mathcal{B}_N(m'_i))$  on  $\mathcal{B}_N$  induces a natural periodic hierarchy. For the next construction to make sense it is important to recall the construction of  $T_{aux}$ .

Given a 1-minimal measure  $m'_i$  on a  $\lambda_1$ -special band complex  $\mathcal{C}_i$ , we will construct a tree  $T^{aux}(\mathcal{C}_i, m'_i, \lambda_1)$  we shall construct this tree one edge at a time. We will still use Convention 4.63.

1. The root of the tree will be  $\mathcal{C}_i$ . We draw an auxiliary edge from  $\mathcal{C}_i$  to  $\mathcal{B}_1$  where  $\mathcal{B}_1$  is the corresponding band complex in periodic block form. We say that  $\mathcal{B}_1$  is at level  $\lambda$ . If  $J(\lambda)$  is maximal then we stop.
2. Suppose we have constructed  $n$  edges of the tree already and that the last vertex that was added is  $\mathcal{B}_n$ . We have three separate cases.



- (a) If  $\mathcal{B}_n$  is at level  $(\lambda, \lambda_{i(1)}, \dots, \lambda_{i(n)})$  and is in periodic block form and one of periodic blocks  $J(\lambda), \dots, J(\lambda_{i(n)})$ , say  $J(\lambda_{i(j)})$ , is maximal; then we backtrack to the  $\lambda_{i(j)}$ -special vertex  $\mathcal{B}'_{i(j)}$ . Now  $\mathcal{B}'_{i(j)}$  has a 1-minimal measure induced from  $m'_i$  and  $\mathcal{C}_i$ , so we do the restricted elimination process until  $\lambda_{i(j)}$  is no longer the carrier. Suppose this takes  $k$  entire transformations then we have add the vertices  $\mathcal{B}_{n+1}, \dots, \mathcal{B}_{n+k}$  to our tree. Where  $\mathcal{B}_{n+1}$  is connected to  $\mathcal{B}'_{i(j)}$  by a *regular* edge and the same is true for  $\mathcal{B}_{n+i}, \mathcal{B}_{n+i+1}$  where  $i+1 \leq k$ . Also we had that  $\mathcal{B}'_{i(j)}$  was at level  $(\lambda, \lambda_{i(1)}, \dots, \lambda_{i(j-1)})$ , and the same is true for these  $k$  new vertices.

Consider the upper bound for  $p(\mathcal{B}_n, \lambda_{i(j)})$  given by Corollary 4.62. This bounds the the number of times in a row  $\lambda_{i(j)}$  will be the carrier base. Suppose this wasn't the case. Then  $\mathcal{B}_n$  has an induced 1-minimal measure  $m'_n$ , which induces a pattern  $\tau$ . Since  $\lambda_{i(j)}$  is a carrier more than  $p(\mathcal{B}_n, \lambda_{i(j)})$  times this means that there must be naked segments of  $J(\lambda_{i(j)})$  that has length more than  $tr(\lambda_{i(j)})$ . We can therefore apply a Dehn twist as seen in Section 4.6.3 to get another track  $\tau'$  in  $\mathcal{B}_n$ , this track is essential and  $Gp(\tau') = \{1\}$ , we also see that the induced measure  $m(\tau')$  is less than  $m'_n$ , contradicting 1-minimality.

- (b) If  $\mathcal{B}_n$  is at level  $(\lambda, \lambda_{i(i)}, \dots, \lambda_{i(n)})$ , is  $\lambda'$ -special and  $\mathcal{B}_{n-1}$  is not  $\lambda'$ -special then  $\mathcal{B}_{n+1}$  is the band complex obtained by bringing  $\mathcal{B}_n$  with the induced 1-minimal measure  $m'_n$  to periodic block form equipped with the induced periodic hierarchy. We connect  $\mathcal{B}_n$  to  $\mathcal{B}_{n+1}$  via an *auxiliary* edge. We say  $\mathcal{B}_{n+1}$  is at level  $(\lambda, \lambda_{i(i)}, \dots, \lambda_{i(n)}, \lambda')$
- (c) If  $\mathcal{B}_n$  is at level  $(\lambda, \lambda_{i(i)}, \dots, \lambda_{i(n)})$  and doesn't fall into one of the previous two cases, then we apply either an entire transformation or a thinning move as required in our restricted elimination process to obtain the band complex  $\mathcal{B}_{n+1}$ , we connect  $\mathcal{B}_n$  to  $\mathcal{B}_{n+1}$  by a normal edge and say that  $\mathcal{B}_{n+1}$  is also at level  $(\lambda, \lambda_{i(i)}, \dots, \lambda_{i(n)})$ .

3. Because the tree  $T^{aux}(\mathcal{C}, m'_i, \lambda)$  is induced by a measure it will be finite, on the other hand the construction only stops once we have a band complex in periodic block form with  $J(\lambda)$  maximal. As explained in (a) above, the bound we get using Corollary 4.62 bounds the number of times in a row  $\lambda$  will be a carrier in the original elimination process.

**Proposition 4.67.** *For any 1-minimal measure  $m$  on a  $\lambda$ -special band complex  $\mathcal{C}_1$ . The finite tree  $T_{aux}(\mathcal{C}_1, \lambda)$  contains the tree  $T^{aux}(\mathcal{C}_1, m, \lambda)$ . In particular  $p(\mathcal{C}_1, \lambda)$  as computed from  $T_{aux}(\mathcal{C}_1, \lambda)$  bounds the number of times in a row  $\lambda$  will be the carrier in the elimination process induced by  $m$ .*

*Proof.* This follows immediately from the construction of  $T_{aux}(\mathcal{C}_1, \lambda)$ ,  $T^{aux}(\mathcal{C}_1, m, \lambda)$ , the 1-minimality assumption and Proposition 4.46.  $\square$

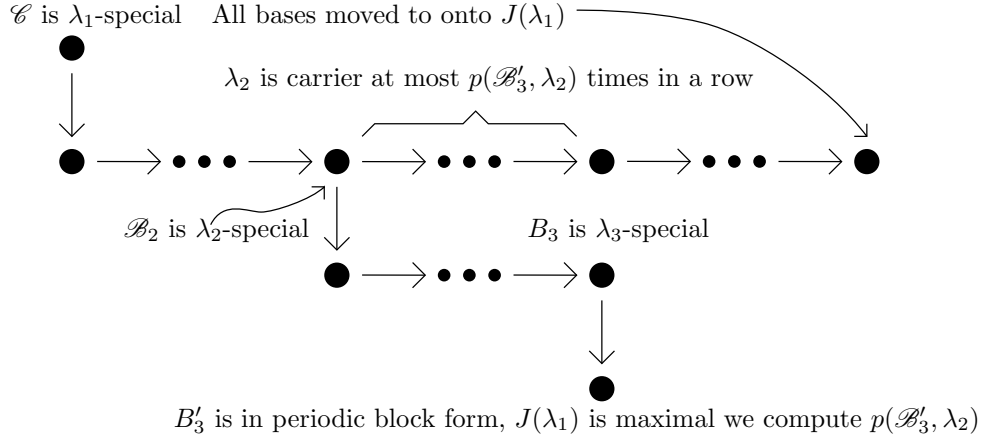


Figure 18: An example of  $T^{aux}(\mathcal{C}, \lambda_1)$ . Auxiliary edges are vertical.

*proof of Theorem 3.1.* This follows from Corollary 3.27, Corollary 3.30, the bounds given in Sections 3.6.1, 3.6.2, Proposition 4.46, and Proposition 4.67.  $\square$

*proof of Theorem 1.1.* Follows from Theorem 3.1  $\square$

## References

- [AHT06] Ian Agol, Joel Hass, and William Thurston. The computational complexity of knot genus and spanning area. *Trans. Amer. Math. Soc.*, 358(9):3821–3850 (electronic), 2006.
- [BF95] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121(2):287–321, 1995.
- [BS88] Joan S. Birman and Caroline Series. Algebraic linearity for an automorphism of a surface group. *J. Pure Appl. Algebra*, 52(3):227–275, 1988.
- [Bul70] V. K. Bulitko. Equations and inequalities in a free group and a free semigroup. *Tul. Gos. Ped. Inst. Učen. Zap. Mat. Kaf.*, (2 Geometr. i Algebra):242–252, 1970.
- [DF05] Guo-An Diao and Mark Feighn. The Grushko decomposition of a finite graph of finite rank free groups: an algorithm. *Geom. Topol.*, 9:1835–1880 (electronic), 2005.
- [DG08] François Dahmani and Daniel Groves. Detecting free splittings in relatively hyperbolic groups. *Trans. Amer. Math. Soc.*, 360(12):6303–6318, 2008.
- [DS99] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.

- [Dun85] M. J. Dunwoody. The accessibility of finitely presented groups. *Invent. Math.*, 81(3):449–457, 1985.
- [GLP94] D. Gaboriau, G. Levitt, and F. Paulin. Pseudogroups of isometries of  $\mathbf{R}$  and Rips’ theorem on free actions on  $\mathbf{R}$ -trees. *Israel J. Math.*, 87(1-3):403–428, 1994.
- [KM05a] Olga Kharlampovich and Alexei Myasnikov. Implicit function theorem over free groups. *J. Algebra*, 290(1):1–203, 2005.
- [KM05b] Olga Kharlampovich and Alexei G. Myasnikov. Effective JSJ decompositions. In *Groups, languages, algorithms*, volume 378 of *Contemp. Math.*, pages 87–212. Amer. Math. Soc., Providence, RI, 2005.
- [Mak82] G. S. Makanin. Equations in a free group. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(6):1199–1273, 1344, 1982.
- [Rau79] Gérard Rauzy. Échanges d’intervalles et transformations induites. *Acta Arith.*, 34(4):315–328, 1979.
- [Vee82] William A. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)*, 115(1):201–242, 1982.